Chapter (1)

Introduction and preliminaries

1.1 Introduction

The classification of finite simple groups was declared accomplished in 1955 through 1983 by Daniel Gorenstein [14] , though some problems surfaced (specifically in the classification of quasithin groups, which were plugged in 2004) . As of 2010, work on improving the proofs and understanding continues .

A solvable group is a group that can be constructed from abelian groups using extensions. If S=Sol(G) is the solvable radical of G (which is a product of solvable normal subgroups of G) then G/S is semi simple . Hence every finite group is an extension of solvable group by a semi-simple group , but any semi-simple group is a direct product of simple groups and all finite simple groups are well known. So it is important to study the structure of the solvable subgroups of large orders in any finite simple group . Here we find out the structures and representations of solvable subgroups of large orders in some non-abelian finite simple groups .

1.2 Preliminaries

Definition1.2.1. A group is a non empty set G with a binary operation defined on it satisfying

(1) $\forall g, h, i \in G, (gh)i = g(hi)$ (associativity)

(2) $\exists\, 1 \in G$ such that $1g = g1 = g, \ \forall g \in G$ (identity)

(3) $\forall g \in G, \ \exists\, g^{-1}$ such that $gg^{-1} = g^{-1}g = 1$ (inverses)

Definition 1.2.2. Let G be a group. We call H a subgroup of G if H is a subset of G and is itself a group under the binary operation inherited from G.

Definition 1.2.3. Let G and H be groups. A homomorphism from G to H is a map $\phi : G \to H$ satisfying

$\phi(gh) = \phi(g)\phi(h) \quad \forall g, h \in G$

If ϕ is a bijection, then ϕ is an isomorphism. We then also call G and H isomorphic, and we write $G \cong H$.

Definition1.2.4. A subgroup N of a group G is called a normal subgroup if

$$Ng = gN \ (\forall g \in G).$$

Definition1.2.5. A maximal subgroup *H* of a group *G* is a proper subgroup, such that no proper subgroup K contains H strictly.

Definition1.2.6. A group G is a simple group if it is a nontrivial group ($G \neq \{e\}$) and if its only normal subgroups are the trivial group and G.

Definition1.2.7. The center Z(G) of a group G is the set of elements in G that commute with every element of G. In other words,

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Definition1.2.8. The centralizer of an element z of a group G is the set of elements of G which commute with z : $C_G(z) = \{x \in G \,, xz = zx\}$.

Definition1.2.9. If x, y ∈ G,we say x is conjugate to y if there exists g ∈ G such that gy=xg . We call the set of all elements conjugate to x the conjugacy class of x, denoted CL(x)= $x^G=\{gxg^{-1} \mid g \in G\}$. The conjugacy classes of a group are disjoint and the union of all the conjugacy classes forms the group.

Definition1.2.10. A group G is called cyclic if there exists an element g in G such that G = <g> = { g^n | n is an integer } ,if g^n =1 and n is the least the integer with this property then G is denoted by C_n .

Definition1.2.11. A dihedral group G is a group generated by two elements x,y such that $G=\langle x,y | x^2 = y^n = (xy)^n = 1\rangle$ and |G|=2n.

Definition1.2.12. Suppose G = NH and N ∩ H = 1 where $N \trianglelefteq G$ and $H \leq G$. Then G is said to be a semidirect product of N and H , $N \rtimes H$ (the split extension of N by H denoted by N : H) .

Definition1.2.13. Let G be a group. For any a,b∈G , the element $a^{-1}b^{-1}ab$ is called the commutator of a and b. The commutator subgroup(also called the derived subgroup,and denoted G′ or $G^{(1)}$) of G it is the subgroup generated by all the commutators.

Definition1.2.14. If G is a group and X is a set, then a (left) group action of G on X is a function $G\times X \to X$, $(g,x) \to g.x$ that satisfies the following two axioms:

Associativity : (gh).x = g.(hx) for all g, h in G and all x in X. (Here, gh denotes the result of applying the group operation of G to the elements g and h.)

Identity : e.x = x for all x in X. (Here, e denotes the neutral element of the group G.)

The set X is called a (left) G-set. The group G is said to act on X (on the left).

Definition1.2.15. A finite group is called a p-group if its order is a power of a prime p.

Definition1.2.16. An elementary abelian group is a finite abelian group, where every nontrivial element has order p, where p is a prime;in particular it is a p-group.

Definition1.2.17. A p-group G is called extra special if its center Z is cyclic of order p, and the quotient G/Z is a non-trivial elementary abelian p-group.

Extra special groups of order p^{1+2n} are often denoted by the symbol p^{1+2n}. For example, 2^{1+24} stands for an extra special group of order 2^{25}.