Kingdom of Saudi Arabia
Ministry of Higher education
Umm Al Qura University
Faculty of Applied Sciences
Department of Mathematical Sciences

(p,q,r)-Generators of Certain Types of Subgroups of large Orders and their Representations in Some Finite Simple Groups

A Thesis submitted to the Faculty of the
Applied Sciences at Umm AlQura University
in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics

by

Abeer Abdullah S. AlGhawazi

Supervised by

Dr. Abdullah A. A. Abduh

1434 - 2013

Chapter (1)

Introduction and preliminaries

1.1 Introduction

The classification of finite simple groups was declared accomplished in 1955 through 1983 by Daniel Gorenstein [14] , though some problems surfaced (specifically in the classification of quasithin groups, which were plugged in 2004) . As of 2010, work on improving the proofs and understanding continues .

A solvable group is a group that can be constructed from abelian groups using extensions. If S=Sol(G) is the solvable radical of G (which is a product of solvable normal subgroups of G) then G/S is semi simple . Hence every finite group is an extension of solvable group by a semi-simple group , but any semi-simple group is a direct product of simple groups and all finite simple groups are well known. So it is important to study the structure of the solvable subgroups of large orders in any finite simple group . Here we find out the structures and representations of solvable subgroups of large orders in some non-abelian finite simple groups .

1.2 Preliminaries

Definition1.2.1. A group is a non empty set G with a binary operation defined on it satisfying

(1) $\forall g,h,i \in G, (gh)i = g(hi)$ (associativity)

(2) $\exists\ 1 \in G$ such that $1g = g1 = g,\ \forall g \in G$ (identity)

(3) $\forall g \in G,\ \exists\ g^{-1}$ such that $gg^{-1} = g^{-1}g = 1$ (inverses)

Definition 1.2.2. Let G be a group. We call H a subgroup of G if H is a subset of G and is itself a group under the binary operation inherited from G.

Definition 1.2.3. Let G and H be groups. A homomorphism from G to H is a map $\phi : G \to H$ satisfying

$\phi(gh) = \phi(g)\phi(h) \quad \forall g,h \in G$

If ϕ is a bijection, then ϕ is an isomorphism. We then also call G and H isomorphic, and we write $G \cong H$.

Definition1.2.4. A subgroup N of a group G is called a normal subgroup if

$$Ng = gN \ (\forall g \in G).$$

Definition1.2.5. A maximal subgroup *H* of a group *G* is a proper subgroup, such that no proper subgroup K contains H strictly.

Definition1.2.6. A group G is a simple group if it is a nontrivial group ($G \neq \{e\}$) and if its only normal subgroups are the trivial group and G.

Definition1.2.7. The center Z(G) of a group G is the set of elements in G that commute with every element of G. In other words,

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Definition1.2.8. The centralizer of an element z of a group G is the set of elements of G which commute with z : $C_G(z) = \{x \in G, xz = zx\}$.

Definition1.2.9. If x, y ∈ G,we say x is conjugate to y if there exists g ∈ G such that gy=xg . We call the set of all elements conjugate to x the conjugacy class of x, denoted $CL(x)= x^G=\{gxg^{-1} \mid g \in G\}$. The conjugacy classes of a group are disjoint and the union of all the conjugacy classes forms the group.

Definition1.2.10. A group G is called cyclic if there exists an element g in G such that $G = \langle g \rangle = \{ g^n \mid n \text{ is an integer} \}$,if $g^n =1$ and n is the least the integer with this property then G is denoted by C_n .

Definition1.2.11. A dihedral group G is a group generated by two elements x,y such that $G=\langle x,y \mid x^2 = y^n = (xy)^n = 1\rangle$ and |G|=2n.

Definition1.2.12. Suppose G = NH and $N \cap H = 1$ where $N \trianglelefteq G$ and $H \leq G$. Then G is said to be a semidirect product of N and H , $N \rtimes H$ (the split extension of N by H denoted by N : H) .

Definition1.2.13. Let G be a group. For any a,b∈G , the element $a^{-1}b^{-1}ab$ is called the commutator of a and b. The commutator subgroup(also called the derived subgroup,and denoted G' or $G^{(1)}$) of G it is the subgroup generated by all the commutators.

Definition1.2.14. If G is a group and X is a set, then a (left) group action of G on X is a function $G\times X \to X$, $(g,x) \to g.x$ that satisfies the following two axioms:

Associativity : (gh).x = g.(hx) for all g, h in G and all x in X. (Here, gh denotes the result of applying the group operation of G to the elements g and h.)

Identity : e.x = x for all x in X. (Here, e denotes the neutral element of the group G.)

The set X is called a (left) G-set. The group G is said to act on X (on the left).

Definition1.2.15. A finite group is called a p-group if its order is a power of a prime p.

Definition1.2.16. An elementary abelian group is a finite abelian group, where every nontrivial element has order p, where p is a prime;in particular it is a p-group.

Definition1.2.17. A p-group G is called extra special if its center Z is cyclic of order p, and the quotient G/Z is a non-trivial elementary abelian p-group.

Extra special groups of order p^{1+2n} are often denoted by the symbol p^{1+2n}. For example, 2^{1+24} stands for an extra special group of order 2^{25}.

الملخص ...

تكمن أهمية الزمر البسيطة المنتهية في كونها "اللبنة الأساسية" لأي زمرة منتهية مثلها في ذلك مثل الأعداد الأولية التي تعتبر "اللبنة الأساسية" لكافة الأعداد الصحيحة . وقد تم تصنيف الزمر البسيطة المنتهية من قبل العالم دانيل جورنشتاين ومجموعة من العلماء المهتمين خلال الأعوام من 1955-1983 . ومن ثم أصبحت دراسة بنية كل زمرة بسيطة منتهية والبنية الأساسية لزمرها الجزئية وتمثيلاتها ، أصبحت تمثل أهمية في كثير من المسائل البحثية التي ما زال جزء كبير منها قيد البحث العلمي . والزمر الجزئية الأكثر أهمية في تلك المسائل هي العظمى والقابلة للحل والمتلاشية والإبدالية.

وقد اهتم العالم أ. مان [19] بإيجاد الزمر الجزئية القابلة للحل من ذوات الرتب العالية للزمر المتناوبة A_n ، كما اهتم العالم فدوفين [22] بإيجاد الزمر الجزئية الإبدالية ذوات الرتب العالية والزمر الجزئية المتلاشية لجميع الزمر البسيطة المنتهية الموضحة في أطلس الزمر البسيطة المنتهية [23] ، وفي عام 2012 استكمل العالم بريوير ، [3] ، إيجاد رتب وبنية الزمر الجزئية القابلة للحل ذوات الرتب العالية للزمر الترددية البسيطة المنتهية sporadic simple groups.

وحيث ان دراسة بنية الزمر الجزئية القابلة للحل ذوات الرتب العالية من الزمر المنتهية البسيطة يساعد في تصنيف أي من الزمر المنتهية بصورة عامة لان أي زمرة منتهية هي عبارة عن توسعة زمرة قابلة للحل عظمى و زمرة شبه بسيطة وهو من المسائل البحثية المهمة التي ما زال جزء كبير منها قيد البحث العلمي ، فقد ركزنا على نوعية أخرى من الزمر البسيطة المنتهية وهي الزمر الخطية الإسقاطية الخاصة $L_2(p)=PSL_2(p)$ ، حيث p عدد اولي ، والعمل على إيجاد زمرها الجزئية القابلة للحل ذوات الرتب العالية ودراسة بنيتها وتمثيلاتها بالإضافة إلى إيجاد مولداتها الاعتيادية ومولداتها من جداول الصفات Character Tables ، باستخدام أسلوبين أولهما النظري وثانيهما استخدام المعلومات المتاحة في أطلس الزمر البسيطة ونظام الجبر الحاسوبي GAP.

وقد تضمن هذا البحث أربعة أبواب وفقا للآتي ...

الباب الأول ... يتضمن المقدمة والتعاريف والخصائص الأساسية التي استخدمت في هذا البحث .

الباب الثاني ... تضمن بعض النتائج الموضحة في المراجع ومنها إيجاد الزمر الجزئية القابلة للحل من ذوات الرتب العالية للزمر المتناوبة A_n ، وإيجاد رتب وبنية الزمر الجزئية القابلة للحل ذوات الرتب العالية للزمر الترددية البسيطة المنتهية sporadic simple groups ، وقد استخدمنا نظام الجبر الحاسوبي لإيجاد تمثيلات تلك الزمر الجزئية داخل الزمر البسيطة بالإضافة إلى خريطة تسكين صفوف الترافق فيما بينهما The Fusion Map.

الباب الثالث ... ويتضمن النتائج التي حصلنا عليها المتعلقة برتب وبنية الزمر الجزئية القابلة للحل من الرتب العالية للزمر المنتهية البسيطة الغير إبدالية ذوات الرتب اقل من واحد مليون عنصر ، كما أوجدنا مولداتها الاعتيادية ومولداتها من جداول الصفات Character Tables ، وقد استخدمنا في سبيل ذلك نظام الجبر الحاسوبي GAP.

الباب الرابع ... وقد تضمن النتائج الخاصة بالزمر البسيطة المنتهية الغير إبدالية من النوعية الخطية الإسقاطية الخاصة $L_2(p)=PSL_2(p)$ ، حيث p عدد أولي ، وقد تم توضيح الأسلوبين النظري والحاسوبي في كيفية الحصول على تلك النتائج .

Abstract

The finite simple groups are important because in a certain sense they are the "basic building blocks" of all finite groups, somewhat similar to the way prime numbers are the basic building blocks of the integers. The classification of finite simple groups was declared accomplished in 1955 through 1983 by Daniel Gorenstein [14]. After announcement on the classification of finite simple groups investigation of known simple groups becomes one of the most important problem in finite group theory. In particular, subgroups structure of known finite simple group is of interest. The most important subgroups are maximal subgroups, maximal soluble subgroups, maximal nilpotent subgroups, and maximal abelian subgroups. In 2000 Vdovin, E. P. [22] devoted his work to abelian and nilpotent subgroups of maximal order of finite simple groups. In 1986 , Mann, A. [19] , found all solvable subgroups of maximal order in symmetric and alternating groups , and between 2006 to 2012 , Breuer,T., [3] , determined the orders of solvable subgroups of maximal orders in sporadic simple groups and their automorphism groups, using the information in the ATLAS of Finite Groups [23] and the GAP system , he also determined the structures and the conjugacy classes of these solvable subgroups in the big group.

The aim of this work is using the information in the ATLAS of Finite Groups [23] and the GAP computational system , [24], to determine the solvable subgroups of maximal orders in the finite non-abelian simple group $L_2(p)=PSL_2(p)$, the projective special linear groups of dimension 2x2 on GF(p) , and in the finite non-abelian simple groups of orders less than 10^6 . The (p,q,r)-generators , the structures , the character tables and the permutation representations of these subgroups have been found. Also in this work we determine the permutation representations of solvable subgroups of large orders in 20 out of the 26 sporadic simple groups , which have been determined by Breuer,T. , [3] .

Contents

Notations

We follow ATLAS [23] conventions for naming groups, conjugacy classes, characters and representations.

$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
G / H	Factor group
$\cong$	Isomorphism
CL(x)	Conjugacy class of an element x
$C_G(a)$	Centralizer of an element a in a group G
C_p	Cyclic group of order p
D_{2n}	Dihedral group of order 2n
S_n	Symmetric group of degree n
A_n	Alternating group of degree n
$H \rtimes K$	The semi-direct product of H and K
H : K	The split extension of H and K
χ	A character
$1_S \uparrow^G$	The induced Character from a subgroup S to a group G
p : q	The split extension of the cyclic group of order p and the cyclic group of order q which is isomorphic to $C_p \rtimes C_q$.
$L_n(p)=PSL_n(p)$	The projective special linear group of dimension n over GF(p)
$U_n(p)=PSU_n(p)$	The projective special unitary group
$S_n(p)=PSP_n(p)$	The projective symplectic group
Sz(8)	The Suzuki classical group

$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$	The Mathieu's sporadic finite non-abelian simple groups
J_1, J_2, J_3, J_4	The Janko's sporadic finite non-abelian simple groups
Co_1, Co_2, Co_3	The Conway's sporadic finite non-abelian simple groups
$Fi_{22}, Fi_{23}, Fi_{24}$	The Fischer's sporadic finite non-abelian simple groups
HS	The Higman-Sims's sporadic finite non-abelian simple groups
McL	The McLaughlin's sporadic finite non-abelian simple groups
Ru	The Rudvalis sporadic finite non-abelian simple group
O'N	The O'nan's sporadic finite non-abelian simple group
Suz	The Suzuki's sporadic finite non-abelian simple group
He	The Held's sporadic finite non-abelian simple group
HN	The Harada–Norton's sporadic finite non-abelian simple group
Ly	The Lyon's sporadic finite non-abelian simple group
Th	The Thompson's sporadic finite non-abelian simple group
B	The Baby Monistor sporadic finite non-abelian simple group
M	The Monistor sporadic finite non-abelian simple group
p^{1+2n}	The extra special group
p^n	The elementary abelian group
M(S)	The maximal subgroup which contains S
Q_8	The quaternion group (a non-abelian group of order eight)
(p,q,r)-group	That group G generated by two elements , the first one is of order p and the second is of order q and their product is of order r. these two elements can be taken as represenatatives of conjugacy clases from the character table of G