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ON MODULAR INCIDENCE G-ALGEBRAS

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OHOOD AMR ALHARTHI

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Prof. Dr. **AHMAD MOHAMMED ALGHAMDI**

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ملخص الرسالة

عنوان الرسالة:

جبر جي على وحدات الحدوث

الدرجة العلمية:

ماجستير في الرياضيات البحتة (جبر)

اسم الباحث:

عهود عمرو عايض الحارثي

الملخص

في هذه الرسالة ركزنا على دراسة مفهوم جبر الحدوث لمجموعة مرتبة جزئياً على حقل. ثم بعد ذلك أنشأنا مفهوم جبر الحدوث المعياري على حقل ذو مميز p حيث p عدد اولي. قمنا بدراسة تأثير زمرة منتهية على الجبر بشكل عام ثم على جبر الحدوث المعياري بشكل خاص. ثم درسنا الضرب التنسوري لجبر الحدوث. درسنا كتل جبر الحدوث. تتكون الرسالة من اربعة فصول: خصصنا الفصل الاول لدراسة تأثير الزمر ومفهوم الجبرا على حقل وجبر الزمر على حقل. خصصنا الفصل الثاني لدراسة جبر الحدوث ودراسة بعض خصائصه ودراسة جبر الحدوث المعياري وقدمنا بعض الامثلة. في الفصل الثالث تعلمنا الضرب التنسوري للجبر واثبتنا ان الضرب التنسوري لاثنين من جبر الحدوث هو جبر الحدوث. يعتمد الفصل الرابع على دراسة تحليل جبر الحدوث المعياري الى كتل جبر الحدوث ودرسنا الكتل معدومة القوى وقدمنا بعض الأمثلة عليها ودرسنا العلاقة بين جبر الزمر وجبر الحدوث.

Contents

Some Frequently Used Notations

p	prime number
G	finite group
e	identity element of G
X	set
$Stab_G(x)$	stabilizer of x belongs X in G
$O(x)$	orbit of x belongs X
$Fix_X(g)$	fixed point of g in X
$\|G\|$	order of G
$\|X/G\|$	number of orbits
H	subgroup of G
F	field of characteristic p
FG	group algebra of G over F
$End_F(M)$	endomorphism algebra of some module M over F
$Mat(n, F)$	matrix algebra of some positive degree n over F
A	algebra over F
1_A	identity element of A
$J(A)$	Jacobson radical of A
$U(A)$	group of units of A
$Aut(A)$	automorphism group of A
$Z(FG)$	center of FG
$1_{Z(FG)}$	identity element of $Z(FG)$
A^H	set of H-fixed points of A
$Res_H^G : A^G \longrightarrow A^H$	restriction map from A^G to A^H

$t_H^G : A^H \longrightarrow A^G$	relative trace map from H-fixed points to G-fixed points in A
$[G : H]$	index of H in G
$A_H^G = t_H^G(A^H)$	image of a relative trace map
$A(G) = A^G / \sum_{H<G} A_H^G$	Brauer quotient
$Br_H : A^H \longrightarrow A(H)$	Brauer homomorphism on A
$(P, \leq)$	partially ordered set with a binary relation $\leq$
$[x, y]$	interval from x to y
$l[x, y]$	length of $[x, y]$
$I(P, F)$	incidence algebra of P over F
$\delta(x, y)$	kronecker delta function
$\zeta(x, y)$	zeta function
$\lambda(x, y)$	lambda function
$\eta(x, y)$	chain function
$\kappa(x, y)$	cover function
$\mu(x, y)$	möbius function
$\rho(x, y)$	length function
$Z(I(P, F))$	center of an incidence algebra
$1_{Z(I(P,F))}$	identity element of a center of an incidence algebra
$P(A)$	set of points of A
$H_\beta = (H, \beta)$	pointed group
$K_\alpha \leq H_\beta$	K_α is a pointed subgroup of H_β
$Br_\beta : A^H \longrightarrow A(H_\beta)$	quotient map
$H_\beta pr K_\alpha$	H_β is projective relative to K_α
V	vector space
$V \times U$	direct product of two vector spaces V and U
$dimV$	dimension of V
$V \otimes_F U$	tensor product of V and U
$A \otimes_F B$	tensor product of two algebras A and B
$G_1 \times G_2$	direct product of two groups G_1 and G_2
$P_1 \times P_2$	direct product of two posets P_1 and P_2
$\mathcal{C}$	category
$F\mathcal{C}$	category algebra

Abstract

This dissertation is about G-algebra theory and incidence algebras and composed of four parts.

The first part depends on study of the algebra over a field and G-algebra over a field means a finite group acts on certain algebra. This concept is due to A. J. Green. We learn about the concept of an interior G-algebra, introduced by L.Puig and Broué and developed by Külshammer.

The second part is devoted to deal with particular algebra, namely incidence algebra over a field. We study some fundamental properties of an incidence algebra. We then construct a modular incidence algebra. We study the action of a finite group G on a modular incidence algebra.

The third part is based on a joint work with Ahmed Alghamdi. The content of this part is based on a prepublication in which we try to understand in the case of uncountable locally partial order sets. We study tensor product of two incidence algebras. We show that the tensor product of two incidence algebras is an incidence algebra.

The fourth part depends on study of the decomposition of the modular incidence algebra into a block algebra. We study defect groups of incidence algebra, pointed groups and nilpotent blocks. We present some examples of nilpotent blocks.

Keywords: Partially ordered set, Incidence algebras, Incidence functions, Modular incidence algebras, G-poset, Incidence G-algebras, Tensor product, Block incidence algebras, Nilpotent blocks.

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Introduction

Let p be a prime number. A p-modular system is a triple $(K, \mathcal{O}, F)$ where K is a field of characteristic zero, $\mathcal{O}$ is a complete discrete valuation ring and $F = \mathcal{O}/J(\mathcal{O})$ is an algebraically field of characteristic p, where $J(\mathcal{O})$ means a jacobson radical of $\mathcal{O}$. In this dissertation we will depend on the field F.

In Chapter 1, the first section we present the concept of group action. First we shall define a group action G on a set X. We shall define the following notions: stabilizer, orbit and fixed point of a set. We show that the stabilizer is a subgroup of G. We then recast the proof of Orbit-Stabilizer Theorem. We shall recast the proof of Burnside's Counting Theorem. In the second section, we present the concept of algebra A over the field F, which is very important. We will mention some important examples of algebra. Then we shall define G-algebra over the field F. In the third section, after we have presented G-algebra. Assume that G is a finite group and H is a subgroup of G. We will define G-fixed points A^G of algebra A. We will define the restriction map Res^G_H and the relative trace map t^G_H. We will explain the relative trace map is linear and independent of co-set representative. We will prove the image of the relative trace map is an ideal in the subalgebra of G-fixed points of A. After that we will define the Brauer quotient and Brauer homomorphism. We will give some examples.

In mathematics, especially order theory, there is a very useful concept is partially ordered set. It is defined as a set which follows partial order relation. Let P be a set. The relation $\leq$ defined on a set P is known as partially ordered relation and the set P is called a partially ordered set under certain conditions. This is what we study in Chapter 2, Section 2.1. Also in this section we will study an incidence algebra $I(P, F)$ of the locally finite partially ordered set P over the field F. The idea of the incidence algebra of a locally finite partially ordered set was proposed by G.-C.Rota [11] as the basis for a unified study of combinatorial theory. The study of incidence algebras was continued by Smith. We mean by the concept of the incidence algebra $I(P,F)$ as the set of functions mapping intervals of P to F. We will define the addition, the scalar multiplication and another operation which is called convolution product on $I(P,F)$. We will define incidence functions of the incidence algebra $I(P,F)$ and of which the kronecker delta function, the zeta function and the möbius function. Then we explain properties this functions in terms of their reversibility. The most important question, do all functions in the incidence algebra $I(P,F)$ have an inverse?. Are there conditions for this property?. After that we will discuss when $I(P,F)$ and $I(Q,F)$ are isomorphic, does P and Q isomorphic?. This question has been answered by Stanley [4, 13]. We then recast the proof of the Möbius Inversion Formula which can be seen in [11] or [12]. In Section 2.2, defining the field in this section will be more important to us as the type of representation will be determined by the characteristic of the field and its relationship to the order of G. We consider G as a finite group and P is a locally finite poset of subgroups of G ordered by inclusion. We have two cases for the characteristic p, the first if the characteristic p divides the order $|G|$ and in this case the representation is called a modular representation. In the other case if the characteristic p does not divide the order $|G|$ then the modular representation is completely reducible, as with ordinary representations. We study in this section a new concept which is a modular incidence algebra by present some examples. In Section 2.3, we elucidate that a finite group G acts on a poset P. Then we generalize some results to the notion modular incidence G-algebra. Of these results, a finite group G action on a modular incidence algebra. We explain that the modular incidence G-algebra form an interior G-algebra.

In Chapter 3, the concept of tensor products are presented in this chapter. In Section 3.1, if V and U are vector spaces over the same field F, we can define the product of V and U and denoted by $V \times U$. The product $V \times U$ as a vector space over F. We then mention a theorem to depend a tensor product on it and how to construct a tensor product. In Section 3.2, if A and B are algebras over F, we define a tensor product of A and B and denoted by $A \otimes_F B$ which we will explain it is an algebra over F. If we have two finite groups G_1 and G_2. Consider A and B as G_1-algebra and G_2-algebra respectively we conclude that the tensor product $A \otimes_F B$ is a $G_1 \times G_2$-algebra and we prove that the tensor product $A \otimes_F B$ considered an interior $G_1 \times G_2$-algebra. In Section 3.3, if $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are two posets. We present notion of cartesian product of $(P_1, \leq_1)$ and $(P_2, \leq_2)$ and denoted by $(P_1 \times P_2, \leq)$ which is considered a poset. A poset P may be countable or uncountable. We will deal with an uncountable poset. We will mention that if the posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are uncountable then also the cartesian product is uncountable. We can define an incidence algebra of $(P_1 \times P_2, \leq)$ over F and we will learn how to form its elements. If we have two incidence algebras $I(P_1, F)$ and $I(P_2, F)$ then the tensor product $I(P_1, F) \otimes_F I(P_2, F)$ is an incidence algebra. We then prove that the incidence algebra $I(P_1 \times P_2, F)$ is isomorphic to the tensor product of incidence algebra $I(P_1, F) \otimes_F I(P_2, F)$.

In Chapter 4, Section 4.1, since the modular incidence algebra $I(P, F)$ is finite dimensional then we can decomposed it into $e_i I(P, F)$, where e_i is a central primitive idempotent of $I(P, F)$. An algebra $e_i I(P, F)$ is called a block incidence algebra which is an ideal and we prove that. We can decomposed a block incidence algebra into indecomposable $I(P, F)$-module. We define a defect group of a block incidence algebra. In Section 4.2 we will define a concept of a pointed group of the modular incidence algebra and a subgroup of a pointed group. We study some concepts such as a projective relative, a local pointed group, a defect pointed group and a nilpotent block algebra of the modular incidence algebra. We give some examples. In section 4.3 we present a notion of a category $\mathcal{C}$ and give some examples. We define a category algebra $F\mathcal{C}$ of $\mathcal{C}$ over F. Then we connect the two concepts of algebra and incidence algebra together.

We have already written a paper which contains this work in the title: on modular incidence G-algebras. We have already submitted that paper for suitable publication.

Historical Survey

- Gian-Carlo Rota was the first mathematician to study the incidence algebra of locally finite partially ordered sets. In the paper which has the title on the foundations of combinatorial theory I. Theory of Möbius functions published in (1964).
- Richard P.Stanley discussed some results in the incidence algebra. In the paper which has the title structure of incidence algebras and their automorphism groups published in (1970).
- Peter Doubilet, Gian-Carlo Rota and Richard Stanley discussed the main facts on the structure of the incidence algebra of a partially ordered set. In the paper which has the title on the foundations of combinatorial theory (VI): The idea of generating function published in (1972).
- Eugene Spiegel and Christopher J. O'Donnell (1997) studied the incidence algebra and the main topics covered were the maximal and prime

ideals of the incidence algebra, its derivations and isomorphisms, its radicals.

- Ancykutty Joseph studied the incidence algebras and directed graphs. In the paper which has the title on incidence algebras and directed graphs published in (2002).

- Ahmad M. Alghamdi studied the tensor product of incidence algebras. In the paper which has the title tensor product of incidence algebras published in (2014).

Chapter 1

Group action and algebra

In this chapter, we present some basic definitions. We have split this chapter into three sections. In Section 1.1, we present a finite group G acting on a non-empty set X. We develop the main concepts and their properties: stabilizers, orbits, and fixed points. We then recast the proof of Orbit-Stabilizer Theorem and recast the proof of Burnside's Counting Theorem. In Section 1.2, we present the concept of algebra A over a field F. We define a local algebra. We define a group algebra and how to form their elements. We present a finite group G acting on an algebra A which is called G-algebra. We define an interior G-algebra and a block algebra. In Section 1.3, we define a relative trace map and we study its properties. We define a Brauar homomorphism and Brauar construction. We give some examples.

1.1 Group action on a set

In this section, we shall define the most important concepts in algebra, which is a group action on a set. We will define some concepts like stabilizer, orbit and fixed point of a set. We recast the proof of Orbit-Stabilizer Theorem and recast the proof of Burnside's Counting Theorem. We give some examples. For further details and background see [7].

Definition 1.1.1. Let G be a finite group. Let X be a non-empty set. The group G acts on X if there is a function $G \times X \longrightarrow X$ defined by $(g, x) \mapsto gx$ such that $ex = x$ and $g(hx) = (gh)x$ for all $x \in X$, $g, h \in G$. We say that X is a G-set.

Definition 1.1.2. Let X be a non-empty set. Let G be a finite group which acts on X. The stabilizer of $x \in X$ $Stab_G(x) = \{g \in G, gx = x\}$ is a subset of G. The orbit of x is the set $O(x) = \{y \in X, \quad \exists g \in G; \quad gx = y\}$.

Remarks.

- We say that an action is a free action if all stabilizer groups are trivial.
- The orbits $O(x)$ are subsets of X
- If $O(x) = X$ for all $x \in X$, then an action is called a transitive action.

It is very easy to show that the stabilizer of x is a subgroup of G as in the following lemma.

Lemma 1.1.3. Let X be a non-empty set. Let G be a finite group which acts on X. If $x \in X$ then the stabilizer of x is a subgroup of G.

Proof. Note that $Stab_G(x)$ is non-empty since $e \in Stab_G(x)$. Now we want to prove that

- If $f, g \in Stab_G(x)$ then $fg \in Stab_G(x)$.
- If $g \in Stab_G(x)$ then $g^{-1} \in Stab_G(x)$.

If $f, g \in Stab_G(x)$ then $fx = x$ and $gx = x$ so $(fg)x = f(gx) = fx = x$ hence $fg \in Stab_G(x)$. Finally if $g \in Stab_G(x)$ then $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = ex = x$ hence $g^{-1} \in Stab_G(x)$. Therefore $Stab_G(x)$ is a subgroup of G. □

In the following theorem we prove that the number of elements in the orbit $O(x)$ is equal to $[G : Stab_G(x)]$, in another meaning $|O(x)| \cdot |Stab_G(x)| = |G|$.

Theorem 1.1.4. (Orbit-Stabilizer Theorem)
Let G be a finite group acting on a set X and let $x \in X$ then

$$|O(x)| \cdot |Stab_G(x)| = |G|.$$

Proof. We want prove that $|O(x)| \cdot |Stab_G(x)| = |G|$. They can be written following way

$$|O(x)| = \frac{|G|}{|Stab_G(x)|}.$$

By Lagrange's Theorem, if H is a subgroup of G then $|G| = |H|[G : H]$. Since $Stab_G(x)$ is a subgroup of G, so $|G| = |Stab_G(x)|[G : Stab_G(x)]$. Rearranging the terms, we get

$$\frac{|G|}{|Stab_G(x)|} = [G : Stab_G(x)]. \tag{1.1}$$

Therefore, we will prove that $|O(x)| = [G : Stab_G(x)]$. Let

$$\phi : O(x) \longrightarrow \frac{G}{Stab_G(x)}$$

be a map. We will define a map $\phi(y) = gStab_G(x)$ for all $y \in O(x)$. Let $y \in O(x)$ then $\exists g \in G$ such that $gx = y$.

To show that ϕ is injective, suppose that $\phi(y_1) = \phi(y_2)$. So $g_1Stab_G(x) = g_2Stab_G(x)$. Since $y_1, y_2 \in O(x)$ then $g_1x = y_1$ and $g_2x = y_2$. Since $g_1Stab_G(x) = g_2Stab_G(x)$ then $\exists g \in G$ such that $g_2 = g_1g$. Hence $y_2 = g_2x = g_1gx = g_1x = y_1$. Therefore, ϕ is injective.

To show that ϕ is surjective, Let $gStab_G(x)$ be a left coset. If $gx = y$ then $\phi(y) = gStab_G(x)$. Therefore, ϕ is surjective. Hence ϕ is a bijective function and

$$|O(x)| = [G : Stab_G(x)]. \tag{1.2}$$

From (1.1) and (1.2) we get

$$|O(x)| = \frac{|G|}{|Stab_G(x)|}, \quad |O(x)| \cdot |Stab_G(x)| = |G|.$$

□

Definition 1.1.5. Let G be a finite group acts on a set X. For any element $g \in G$ the fixed point of X is $Fix_X(g) = \{x \in X, gx = x\}$.

The Burnside's Counting Theorem asserts the calculate the number of orbits as follows

Theorem 1.1.6. (Burnside's Counting Theorem)
Let X be a finite set. Let G be a finite group that acts on the set X. Then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |Fix_X(g)|,$$

where $|X/G|$ stands for the number of orbits.

Proof. The first step, this can be changed the sum over the group elements $g \in G$ as equivalent sum over the set of element $x \in X$

$$\begin{aligned}\sum_{g\in G} |Fix_X(g)| &= \sum_{g\in G} |\{x \in X : gx = x\}| \\ &= |\{(g, x) : g \in G, x \in X, gx = x\}| \\ &= \sum_{x\in X} |\{g \in G : gx = x\}| \\ &= \sum_{x\in X} |Stab_G(x)|.\end{aligned}$$

Now, use the Orbit-Stabilizer Theorem

$$|Stab_G(x)| = \frac{|G|}{|O(x)|}.$$

So we get

$$\begin{aligned}\sum_{x\in X} |Stab_G(x)| &= \sum_{x\in X} \frac{|G|}{|O(x)|} \\ &= |G| \sum_{x\in X} \frac{1}{|O(x)|}.\end{aligned}$$

Finally, notice that X is the disjoint union of all its orbits in X/G, which means the sum over X may be broken up into separate sums over each individual orbit

$$\sum_{x\in X} \frac{1}{|O(x)|} = \sum_{Y\in X/G} \sum_{x\in Y} \frac{1}{|Y|} = \sum_{Y\in X/G} 1 = |X/G|.$$

So the sum becomes

$$\sum_{x\in X} |Stab_G(x)| = |G| \cdot |X/G|.$$

Hence

$$\sum_{g\in G} |Fix_X(g)| = |G| \cdot |X/G|$$

$$|X/G| = \frac{1}{|G|} \cdot \sum_{g\in G} |Fix_X(g).$$

□

1.2 Algebra and group algebra

In this section, we shall define an algebra A over the field F. We define a group algebra FG over F. We will study a group G acts on an algebra A. We will define an interior G-algebra and a block algebra of group algebra. Most of the result here is in [7, 9, 14] and [15].

Definition 1.2.1. Let p a prime number, let $(K, \mathcal{O}, F)$ be a p-modular system. Let R be a ring, $R \in (K, \mathcal{O}, F)$. An algebra $(A, +, \cdot)$ over the ring R is a set A with sum, product and scalar multiplication such that:

1. $(A, +, \cdot)$ is a ring.
2. $(A, +, \cdot)$ is a vector space over R.
3. $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in R$ and $a, b \in A$.

More than that, the algebra A is free as a R-module. The important examples of algebra are group ring RG of some finite group G over R, the Endomorphism algebra $End_R(M)$ of some module M over R, the matrix algebra $Mat(n, R)$ of some positive degree n over R and the incidence algebra.

An algebra A contains three types of elements are unit, nilpotent and idempotent element. We will explain them in the following definitions.

In the following, let F be an algebraically closed field which has characteristic prime number p and A be an algebra over F.

Definition 1.2.2. A unit element in A over F is an element $x \in A$ such that $xy = 1_A$ for some element $y \in A$.

Definition 1.2.3. An element $b \in A$ is called **nilpotent element** if there exist positive integer n such that $b^n = 0$.

Definition 1.2.4. An idempotent element in an algebra A over F is an element a such that $a^2 = a$.

Definition 1.2.5. The set of elements which is commute with every element of A is called **center** of A and denoted by $Z(A)$. We can write

$$Z(A) = \{a \in A : ab = ba \quad \forall b \in A\}.$$

Definition 1.2.6. Two idempotents a and b in A are called **orthogonal** if $ab = ba = 0$.

Definition 1.2.7. A non-zero idempotent in A is called **primitive** in A if it cannot be written as the sum of two orthogonal non-zero idempotent in A.

Definition 1.2.8. An idempotent a in A is called **a central idempotent** if $a \in Z(A)$.

Definition 1.2.9. The Jacobson radical of A is an unique maximal nilpotent ideal and denote by $J(A)$.

Remarks.

- Certainly the elements 0_A and 1_A are idempotents in A.
- The unit element of A form a group and called the unit group of A and denote by $U(A)$.
- If a is any idempotent in A then so is $b = 1 - a$ moreover a and b are orthogonal.

Definition 1.2.10. An algebra A is a local algebra if it has any one of the following equivalent properties:

- A has an unique maximal ideal.
- A has only 0_A and 1_A as idempotents.
- Every element of A is either unit or nilpotent.
- $A/J(A) \cong F$.
- $A = J(A) \dot{\cup} U(A)$.

Definition 1.2.11. Let G be a finite group and F be the field. The algebra FG is called group algebra over F and its elements are written as follows

$$FG = \{\textstyle\sum_{g \in G} \alpha_g g; \quad \alpha_g \in F, \quad \forall g \in G\}.$$

The basis of FG is the set of elements of G.

Example 1.2.12. Consider $G = S_3$ be the symmetric group and $F = \mathbb{Z}_2$ is the field which has characteristic two. The group algebra

$$\mathbb{Z}_2 S_3 = \{\sum_{g \in S_3} \alpha_g g; \quad \alpha_g \in \mathbb{Z}_2, \quad g \in S_3\}.$$

The number of elements in $\mathbb{Z}_2 S_3$ is 64. We have
$U(\mathbb{Z}_2 S_3) = \{(1), (123), (132), (12), (13), (23), (1) + (123) + (132) + (12) + (13), (1) + (123) + (132) + (12) + (23), (1) + (123) + (132) + (13) + (23), (1) + (123) + (12) + (13) + (23), (1) + (132) + (12) + (13) + (23), (123) + (132) + (12) + (13) + (23)\}$.

The idempotents are $I(\mathbb{Z}_2S_3) = \{0, (1), (123)+(132), (1)+(123)+(132), (132) + (12) + (23), (132) + (12) + (13), (132) + (13) + (23), (123) + (13) + (23), (123) + (12) + (13), (123) + (12) + (23), (1) + (123) + (12) + (13), (1) + (132) + (12) + (13), (1) + (123) + (12) + (23), (1) + (132) + (12) + (23), (1) + (123) + (13) + (23), (1) + (132) + (13) + (23)\}$.

The nilpotents are $N(\mathbb{Z}_2S_3) = \{0, (1)+(12), (1)+(13), (1)+(23), (123)+(132)+(12)+(13), (123)+(132)+(12)+(23), (123)+(132)+(13)+(23), (1)+(123) + (132) + (12) + (13) + (23)\}$.

The central idempotents are $\{0, (1), (123) + (132), (1) + (123) + (132)\}$.

The primitive idempotents are $\{(1) + (123) + (132), (132) + (12) + (23), (132) + (13) + (23), (132) + (12) + (13), (123) + (13) + (23), (123) + (12) + (13), (123)+(12)+(23), (1)+(123)+(12)+(13), (1)+(132)+(12)+(13), (1)+(123) + (12) + (23), (1) + (132) + (12) + (23), (1) + (123) + (13) + (23), (1) + (132) + (13) + (23)\}$.

The central primitive idempotent $\{(123) + (132), (1) + (123) + (132)\}$.

Example 1.2.13. The group algebra $A = \mathbb{Z}_2V_4$ where it is the group algebra of the group V_4 over the field $\mathbb{Z}_2$ which has characteristic two, the number of elements in $\mathbb{Z}_2V_4$ is 16.
$\mathbb{Z}_2V_4 = \{0, 1, a, b, c, 1+a, 1+b, 1+c, a+b, a+c, b+c, 1+a+b, 1+a+c, 1+b+c, a+b+c, 1+a+b+c\}$.
The group algebra $\mathbb{Z}_2V_4$ contains only two idempotents $0, 1$.
$U(\mathbb{Z}_2V_4) = \{1, a, b, c, 1 + a + b, 1 + a + c, 1 + b + c, a + b + c\}$.
$N(\mathbb{Z}_2V_4) = \{0, 1 + a, 1 + b, 1 + c, a + b, a + c, b + c, 1 + a + b + c\}$.
$J(\mathbb{Z}_2V_4) = \{0, 1 + a, 1 + b, 1 + c, a + b, a + c, b + c, 1 + a + b + c\}$.
Note that $\mathbb{Z}_2V_4/J(\mathbb{Z}_2V_4)$ is isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_2V_4$ is the disjoint union of $J(\mathbb{Z}_2V_4)$ and $U(\mathbb{Z}_2V_4)$. Thus the group algebra $\mathbb{Z}_2V_4$ is the local algebra.

In the following definition, we define an important concept in algebra theory, which is a G-algebra, which A.J.Green discovered

Definition 1.2.14. Let G be a finite group and A be an algebra over F. A G-algebra over F is a pair (A, ϕ) where $\phi : G \longrightarrow Aut(A)$ is a group homomorphism and $Aut(A)$ is the group of F-algebra automorphism of A. The action of G on A is given by $a^g = \phi(g)(a)$ where $g \in G$ and $a \in A$. So for $g, h \in G, a, b \in A$ and $\alpha \in F$ we have

$$\begin{aligned} a^e &= a \\ (a^h)^g &= a^{hg} \\ (a+b)^g &= a^g + b^g \\ (ab)^g &= (a^g)(b^g) \\ (\alpha a)^g &= \alpha(a^g), \end{aligned}$$

Theorem 1.2.15. Let G be a finite group. Let A be an algebra over F. Then a function $\phi : G \longrightarrow Aut(A)$ is a group action on A if and only if the function $h : G \longrightarrow U(A)$ is a group homomorphism, $h(g) = \rho_g$ where $\rho_g : A \longrightarrow A$ is a bijective function and $\rho_g(a) = \phi(g)(a) = a^g$.

Proof. Suppose that $\phi : G \longrightarrow Aut(A)$ is a group action on A. We will prove that h is a homomorphism. For $f, g \in G$ by definition $h(fg) = \rho_{fg}$ also $h(f)h(g) = \rho_f \rho_g$. Since ϕ is a group action it follows that for all $a \in A$, $\rho_{fg}(a) = a^{fg} = (a^f)^g = (\rho_f(a))^g = \rho_f(a^g) = \rho_f \rho_g(a)$. So $\rho_{fg} = \rho_f \rho_g$. Hence $h(fg) = h(f)h(g)$. Therefore h is a group homomorphism.
Conversely, suppose that h is a homomorphism. We will prove that ϕ is a group action. Since h is a group homomorphism then $h(fg) = h(f)h(g)$ it follows that $\rho_{fg} = \rho_f \rho_g$. For $a, b \in A$, $f, g \in G$ and $\alpha \in F$ we have

- $a^e = \rho_e(a) = a$
- $(a^f)^g = (\rho_f(a))^g = \rho_f(a^g) = \rho_f \rho_g(a) = \rho_{fg}(a) = a^{fg}$
- $(a + b)^g = \rho_g(a + b) = \rho_g(a) + \rho_g(b) = a^g + b^g$
- $(ab)^g = \rho_g(ab) = \rho_g(a)\rho_g(b) = a^g b^g$
- $(\alpha a)^g = \rho_g(\alpha a) = \alpha \rho_g(a) = \alpha a^g$.

□

Example 1.2.16. Let G be a finite group. Let A be a G-algebra over F. Consider the set of all idempotents of A and denoted by X. We define an action of G on X by conjugation.

In the following lemma, we show that G acts on X.

Lemma 1.2.17. Let G be a finite group. Let X be a set of all idempotents of A, where A is a G-Algebra over F. Then G acts on X by conjugation.

Proof. If $g \in G$ and $x \in X$ then $x * g = g^{-1}xg$.
So $(g^{-1}xg)^2 = g^{-1}xgg^{-1}xg = g^{-1}x^2g = g^{-1}xg$ hence $g^{-1}xg \in X$. Now we check of the conditions of the action. Firstly if $x \in X$ then $x * e = e^{-1}xe = x$. Secondly if $g, h \in G$ and $x \in X$ then $x*(g*h) = (gh)^{-1}x(gh) = g^{-1}h^{-1}xgh = h^{-1}g^{-1}xgh = h^{-1}(x * g)h = (x * g) * h$. Furthermore G acts on X by conjugation. □

The stabilizer of $x \in X$ is a subgroup of G where

$$\begin{aligned} Stab_G(x) &= \{g \in G, \quad x * g = x\} \\ &= \{g \in G, \quad g^{-1}xg = x\} \\ &= \{g \in G, \quad xg = gx\}. \end{aligned}$$

The orbit of $x \in X$ is a subset of X where

$$\begin{aligned} O(x) &= \{f \in X, \quad \exists g \in G; \quad x * g = f\} \\ &= \{f \in X, \quad \exists g \in G; \quad g^{-1}xg = f\}. \end{aligned}$$

The fixed point of $g \in G$ is a subset of X as a notation

$$\begin{aligned} Fix_X(g) &= \{x \in X, \quad x * g = x\} \\ &= \{x \in X, \quad g^{-1}xg = x\}. \end{aligned}$$

Remark. The unit group $U(A)$ of an algebra A acts on the set of all idempotents of A.

Definition 1.2.18. Let G be a finite group and A be an algebra over F. The algebra A is called interior G-algebra over F if there is a pair (A, ψ) where $\psi : G \longrightarrow U(A)$ is a homomorphism of group.

Theorem 1.2.19. Every interior G-algebra is a G-algebra.

Proof. Let G be a finite group. Let A be an interior G- algebra. Then there is a group homomorphism $\psi : G \longrightarrow U(A)$. From Theorem 1.2.15, produces that there exist a group action on A. Hence A is a G-algebra. □

Example 1.2.20. Let $A = FG$ be a group algebra over F. Then FG is an interior G-algebra since there is a group homomorphism $\rho : G \longrightarrow U(A)$ such that $\rho(x) = x$ via inclusion map sending $x \in G$ to $x \in G \subset U(A)$.

Definition 1.2.21. Let G be a finite group. Let M be a FG-module. The endomorphism of M is a homomorphism $f : M \longrightarrow M$.

Example 1.2.22. Let G be a finite group. Let M be a FG-module. The endomorphism algebra $(End_F(M), +, \circ)$ is an interior G-algebra since $End_F(M)$ is an F-algebra then there is a group homomorphism $\rho : G \longrightarrow GL(M)$.

Definition 1.2.23. Let G be a finite group. Let M be a FG-module and N be a submodule of M. We say that N is a direct summand of M if there exist other submodule N' of M such that $M = N \oplus N'$.

Definition 1.2.24. Let G be a finite group and let $A = FG$ be a group algebra of G over F. Consider the central primitive idempotent e_i. This means that $e_i^2 = e_i$. So, arises a structure $e_iFG = e_iFGe_i = FGe_i$ and denote by $B = FGe_i$, $1 \leq i \leq t$. The algebra B over F is called a block algebra of FG and e_i is called a block idempotent of FG.
Since $e_i \in Z(FG)$. Then we have

$$1_{Z(FG)} = e_1 + e_2 + ... + e_t$$

and we have

$$FG = B_1 \oplus B_2 \oplus ... \oplus B_t.$$

We note that the block algebra B_i as a direct summand of FG.

Example 1.2.25. Consider $A = \mathbb{Z}_2S_3$ be the group algebra over $\mathbb{Z}_2$. The central primitive idempotents of A are $e_1 = (123) + (132)$ and $e_2 = (1) + (123) + (132)$. Thus we have two blocks in A, $B_1 = e_1A$ and $B_2 = e_2A$.

$B_1 = \{0, (123)+(132), (1)+(132), (1)+(123), (23)+(13), (12)+(23), (12)+(13), (123) + (132) + (13) + (23), (123) + (132) + (12) + (23), (123) + (132) + (12)+(13), (1)+(132)+(13)+(23), (1)+(132)+(12)+(23), (1)+(132)+(12)+(13), (1)+(123)+(13)+(23), (1)+(123)+(12)+(23), (1)+(123)+(12)+(13)\}$

$B_2 = \{0, (1)+(123)+(132), (12)+(13)+(23), (1)+(123)+(132)+(12)+(13) + (23)\}$.

We note that $1_{Z(A)} = e_1 + e_2$ and $A = B_1 \oplus B_2$.

1.3 Relative trace map

In this section, we shall define on fixed points of algebra A over a field which has characteristic $p > 0$, where p is a fixed prime number. We will define a relative trace map and a Brauer homomorphism. We will prove the image of a trace map is an ideal of a fixed point. Most of the result here is in [9, 10, 14] and [15].

Definition 1.3.1. Let G be a finite group. Let A be a G-algebra over F. Let H be a subgroup of G. The set of H-fixed points of A is written

$$A^H = \{a \in A, a^h = a, \text{ for all } h \in H\}.$$

The set of G-fixed points of A can be written as

$$A^G = \{a \in A, a^g = a, \text{ for all } g \in G\}.$$

We define a map $Res_H^G : A^G \longrightarrow A^H$ by $Res_H^G(x) = x$ for all $x \in A^G$. Here $A^G \subseteq A^H$. The map Res_H^G is called the inclusion or restriction map from A^G to A^H.
Now we define the reverse direction map for the restriction map. The map $t_H^G : A^H \longrightarrow A^G$ defined by

$$t_H^G(a) = \sum_{g \in T} a^g,$$

for all $a \in A^H$. Where T to be a co-set representative of H in G. The map t_H^G is called the relative trace map from H-fixed points to G-fixed points in A.

Remark. For any $g \in G$ we have $(A^H)^g = A^{g^{-1}Hg}$ while if $g \in N_G(H)$ then we have $(A^H)^g = A^H$ hence we can consider A^H as an $N_G(H)$-algebra over F.

Example 1.3.2. Let G be a finite group. Let FG be a group algebra over F. Let H be a subgroup of G. The fixed point of the conjugation action of G on FG is equal the center of group algebra FG, and the fixed point of the conjugation action of H on FG is equal the centralizer of H of group algebra FG. Then the relative trace map is defined to be

$$t_H^G : C_{FG}(H) \longrightarrow Z(FG).$$

Example 1.3.3. Let G be a finite group. Let H be a subgroup of G. Let M be a FG-module. Let $End_F(M)$ be an endomorphism algebra over F. The group G acts on $End_F(M)$ by $f^g(m) = gf(g^{-1}m)$ for all $f \in End_F(M)$ and $g \in G$. We have $A^G = End_{FG}(M)$ and $A^H = End_{FH}(M)$. Then the relative trace map is defined to be

$$t_H^G : End_{FH}(M) \longrightarrow End_{FG}(M),$$

$$t_H^G(f) = \sum_{t \in T} f^t, \quad f \in End_{FH}(M).$$

Theorem 1.3.4. Let G be a finite group and H be a subgroup of G. Let A be a G-algebra over F. Let $t_H^G : A^H \longrightarrow A^G$ be a relative trace map from H-fixed points to G-fixed points in A. Then

1. $t_H^G(a) \in A^G$ for all $a \in A^H$.

2. t_H^G is linear.

3. t_H^G is independent of co-set representative.

Proof. If G is a finite group. If A is a G-algebra over F and H is a subgroup of G. If $t_H^G : A^H \longrightarrow A^G$ is a relative trace map and defined by

$$t_H^G(a) = \sum_{g \in T} a^g, \quad \forall a \in A^H,$$

where T to be a co-set representative of H in G.

1. If H has finite index in G denoted by $|G : H| = n$. Thus the left cosets of H in G are $eH, g_2H, g_3H, ..., g_nH$. So $T = \{e, t_1, t_2, ..., t_n\}$. we have $Tg = \{tg : t \in T\}$ is a co-set representative.

$$\begin{aligned}(t_H^G(a))^g &= (\sum_{t \in T} a^t)^g \\ &= \sum_{t \in T} a^{tg} \\ &= t_H^G(a).\end{aligned}$$

So, $t_H^G(a) \in A^G$ for all $a \in A^H$.

2. Let $a, b \in A^H$

$$\begin{aligned} t_H^G(a+b) &= \sum_{t \in T}(a+b)^t \\ &= \sum_{t \in T}(a^t + b^t) \\ &= \sum_{t \in T} a^t + \sum_{t \in T} b^t \\ &= t_H^G(a) + t_H^G(b). \end{aligned}$$

Let $a \in A^H$ and $\alpha \in F$

$$\begin{aligned} t_H^G(\alpha a) &= \sum_{t \in T}(\alpha a)^t \\ &= \sum_{t \in T} \alpha(a)^t \\ &= \alpha \sum_{t \in T}(a)^t \\ &= \alpha t_H^G(a). \end{aligned}$$

Hence $t_H^G(a)$ is a linear map.

3. Let T and M be two co-set representatives. Where $T = \{t_1, t_2, ..., t_n\}$, $M = \{m_1, m_2, ..., m_n\}$. Then for any $t_i \in T, \exists\ m_j \in M$ such that $t_i \in MH$. So $t_i = m_j h$, $h \in H$

$$\begin{aligned} t_H^G(a) &= \sum_{t_i \in T}(a)^{t_i} \\ &= \sum_{m_j h \in T}(a)^{m_j h} \\ &= \sum_{m_j h \in T}(a^h)^{m_j} \\ &= \sum_{m_j \in M}(a)^{m_j}. \end{aligned}$$

Hence $t_H^G(a)$ is independent of co-set representative of H in G. □

Theorem 1.3.5. Let G be a finite group and A be a G-algebra over F. The image of the relative trace map is an ideal in the subalgebra of G-fixed points of A.

Proof. Suppose H is a subgroup of G. The image of t_H^G can be written as $\text{Image}(t_H^G) = \{a \in A^G : \exists b \in A^H \text{ such that } t_H^G(b) = a\}$. Clearly $\text{Image}(t_H^G) \subseteq A^G$.
We need prove that $\text{Image}(t_H^G)$ is both a left ideal and a right ideal. First we prove that $\text{Image}(t_H^G)$ is a subring of A^G. Since $0 \in A^G$ and $0 \in A^H$ we have $0 = \sum_{t\in T} 0^t = t_H^G(0) \in \text{Image}(t_H^G)$. Hence $\text{Image}(t_H^G) \neq \phi$.
Let $a_1, a_2 \in \text{Image}(t_H^G)$ so there are $b_1, b_2 \in A^H$. Thus

$$\begin{aligned} a_1 - a_2 &= t_H^G(b_1) - t_H^G(b_2) \\ &= \sum_{t\in T} b_1^t - \sum_{t\in T} b_2^t \\ &= \sum_{t\in T}(b_1^t - b_2^t) \\ &= \sum_{t\in T}(b_1 - b_2)^t \\ &= t_H^G(b_1 - b_2). \end{aligned}$$

Where $b_1 - b_2 \in A^H$. Therefore $a_1 - a_2 \in \text{Image}(t_H^G)$. Hence $\text{Image}(t_H^G)$ is a subring of A^G.
Second, we prove that xa, ax belong to $\text{Image}(t_H^G)$. Let $a \in \text{Image}(t_H^G)$, $x \in A^G$. So, there is $b \in A^H$ such that $t_H^G(b) = a$

$$\begin{aligned} x \cdot a &= x \cdot t_H^G(b) \\ &= x \cdot \sum_{t\in T} b^t \\ &= \sum_{t\in T} xb^t \\ &= \sum_{t\in T} x^t b^t \\ &= \sum_{t\in T} (xb)^t \\ &= t_H^G(xb). \end{aligned}$$

Where $x \in A^G \subseteq A^H$ so $xb \in A^H$. Therefore $xa \in \text{Image}(t_H^G)$ and

$$\begin{aligned} a \cdot x &= t_H^G(b) \cdot x \\ &= \sum_{t \in T} b^t \cdot x \\ &= \sum_{t \in T} b^t x^t \\ &= \sum_{t \in T} (bx)^t \\ &= t_H^G(bx), \quad bx \in A^H. \end{aligned}$$

Therefore $ax \in \text{Image}(t_H^G)$. Hence $\text{Image}(t_H^G)$ is an ideal in A^G. □

Now we set $A_H^G = t_H^G(A^H)$ and we proved A_H^G ia an ideal of A^G. We define the Brauer quotient

$$A(G) = A^G / \sum_{H<G} A_H^G.$$

Definition 1.3.6. Let G be a finite group. Let A be a G-algebra over F. Let H be a subgroup of G. Let $A(H)$ be a Brauer quotient. The map $Br_H : A^H \longrightarrow A(H)$ defined by $a \longrightarrow a + A_{<H}^H$ is called the Brauer homomorphism on A with respect to H.

Remark. In our case when F has prime characteristic p. For any subgroup H of G and any subgroup K of H, such that $[H : K]$ is not divisible by p, thus $A_K^H = A^H$ and $A(H) = 0$ unless H is a p-group.

Example 1.3.7. The group algebra $A = \mathbb{Z}_2 V_4$ over the field $\mathbb{Z}_2$. We choose a subgroup H of the group $G = V_4$, $H = \langle a \rangle = \{e, a\}$ the index of H in V_4 is equal to 2.
$e \cdot H = \{e, a\}$, $b \cdot H = \{b, c\}$. The coset representative of H in V_4 is $T = \{e, b\}$.
$A^G = Z(\mathbb{Z}_2 V_4) = \mathbb{Z}_2 V_4$ and $A^H = C_{\mathbb{Z}_2 V_4}(H) = \mathbb{Z}_2 V_4$.
Thus $A^G = \mathbb{Z}_2 V_4 = A^H$. The relative trace map is

$$t_H^G : A^H \longrightarrow A^G,$$

$$t_H^G : \mathbb{Z}_2 V_4 \longrightarrow \mathbb{Z}_2 V_4,$$

$$t_H^G(a) = \sum_{t \in T} a^t, \qquad a \in A^H.$$

We get $A_H^G = \text{Image}(t_H^G) = \{0\}$. The Brauer quotient $A(G) = A^G$ and the Brauer homomorphism is

$$Br_G : A^G \longrightarrow A(G),$$

$$Br_G : \mathbb{Z}_2V_4 \longrightarrow \mathbb{Z}_2V_4,$$
$$a \longrightarrow a + \sum_{H<G} A_H^G, \quad a \longrightarrow a.$$

Example 1.3.8. Let S_3 be the symmetric group and $\mathbb{Z}_2$ be the field which has characteristic two. We take $H = \langle 12 \rangle$ to be a subgroup of S_3, the index of H in G is equal to 3. The left cosets of H in S_3 are
$(1) \cdot H = H = \{(1), (12)\}$
$(123) \cdot H = \{(123), (13)\}$
$(132) \cdot H = \{(132), (23)\}$.
The coset representative of H in S_3 is $T = \{(1), (123), (132)\}$. We have $A = \mathbb{Z}_2S_3$ is the group algebra over the field $\mathbb{Z}_2$. We know that its elements are written like this

$$A = \{\sum_{g\in G} \alpha_g g; \quad \alpha_g \in \mathbb{Z}_2, \quad g \in S_3\}.$$

The number of elements in A is $64 = 2^6$.
$A^G = Z(\mathbb{Z}_2S_3) = \{0, (1), (123) + (132), (12) + (13) + (23), (1) + (123) + (132), (1) + (12) + (13) + (23), (123) + (132) + (12) + (13) + (23), (1) + (123) + (132) + (12) + (13) + (23)\}$.
$A^H = C_{\mathbb{Z}_2S_3}(H) = \{0, (1), (12), (1) + (12), (123) + (132), (13) + (23), (1) + (123) + (132), (1) + (13) + (23), (123) + (132) + (12), (12) + (13) + (23), (1) + (123) + (132) + (12), (123) + (132) + (13) + (23), (1) + (12) + (13) + (23), (1) + (123) + (132) + (13) + (23), (123) + (132) + (12) + (13) + (23), (1) + (123) + (132) + (12) + (13) + (23)\}$.
The relative trace map is

$$t_H^G : C_{\mathbb{Z}_2S_3}(H) \longrightarrow Z(\mathbb{Z}_2S_3)$$
$$t_H^G(a) = \sum_{t\in T} a^t, \qquad a \in C_{\mathbb{Z}_2S_3}(H).$$

$\text{Image}(t_H^G) = \{0, (1), (123) + (132), (12) + (13) + (23), (1) + (123) + (132), (1) + (12) + (13) + (23), (123) + (132) + (12) + (13) + (23), (1) + (123) + (132) + (12) + (13) + (23)\}$.
Note that $\text{Image}(t_H^G) = A^G$ and $[S_3 : H] = 3$ is not divisible by 2. Thus $A(G) = 0$.

Chapter 2

Incidence algebras

Our aim in this chapter is to study a new type of algebra which is called incidence algebra which is denoted by $I(P, F)$. In Section 2.1, we define a partially ordered set P and we present some examples of a partially ordered set. If F is a field of characteristic $p > 0$, we can define an incidence algebra $I(P, F)$ of P over F which consists of functions from the cartesian product $P \times P = P^2$ to the field F. We call it the incidence functions. We then study Stanley's Theorem. We define some incidence functions. We then recast the proof of Möbius Inversion Formula. In Section 2.2, we consider G as a finite group. We study the incidence algebra of P over F, where P is a locally finite partially ordered set ordered by inclusion, its elements are subgroups of G. We have two cases: the first case that p divides the order of G and the second that p does not divide the order of G. Through the examples, we will see that either of these two cases the representation will be modular. We then say that an incidence algebra is a modular incidence algebra. In Section 2.3, we define a finite group G acting on a poset P. The show that some posets are receive the action, not all. Then we define a group action on a modular incidence algebra. We define a modular incidence algebra as an interior G-algebra.

2.1 Incidence algebra and incidence functions

In this section, we shall define a partially ordered set P. We study an incidence algebra $I(P, F)$ of P over F. We will study the Stanley's Theorem. We will define incidence functions of the incidence algebra $I(P, F)$. We recast the proof of the Möbius Inversion Formula. All facts and results in this section can be found in [1, 4, 8, 11, 12] and [13].

Definition 2.1.1. A set P with a binary relation $\leq$ is a partially ordered set, often called a poset for short; if it satisfies the following three properties:

- Reflexive: $x \leq x$, for all $x \in P$.
- Anti-symmetric: if $x \leq y$ and $y \leq x$ then $x = y$, for all $x, y \in P$.
- Transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$, for all $x, y, z \in P$.

We denote a partially ordered set by $(P, \leq)$.

Example 2.1.2. The set of natural numbers with the usual order relation $(\mathbb{N}, \leq)$ form a poset. To see that this is a poset

- For all $x \in \mathbb{N}$ then $x \leq x$. Hence the relation $\leq$ is reflexive.
- If $x \leq y$ and $y \leq x$ then $x = y$, for all $x, y \in \mathbb{N}$. Hence the relation $\leq$ is antisymmetric.
- If $x \leq y$ and $y \leq z$ then $x \leq z$, for all $x, y, z \in \mathbb{N}$. Hence the relation $\leq$ is transitive.

Example 2.1.3. The set of natural numbers $\mathbb{N}$ under divisibility mean that $a \leq b$ if a divides b. We take $P = \{2, 4, 8, 16, ...\}$. Then $(P, |)$ is the poset, where $|$ means divides, to check that

- For all $a \in P$ then $a|a$. Hence the relation $|$ is reflexive.
- If $a|b$ and $b|a$ then $a = b$, for all $a, b \in P$. Hence the relation $|$ is antisymmetric.
- If $a|b$ and $b|c$ then $a|c$, for all $a, b, c \in P$. Hence the relation $|$ is transitive.

Example 2.1.4. Take $P = \big\{\{1\}, \{1, 3\}, \{1, 3, 5\}, \{1, 3, 5, 9\}\big\}$ to be a set of subsets of the power set of $\{1, 3, 5, 9\}$. Then $(P, \subseteq)$ the set P with a binary relation $\subseteq$ is the poset. To verify this is the poset

- For $A \in P$ then $A \subseteq A$. Hence the relation $\subseteq$ is reflexive.
- If $A \subseteq B$ then $B \nsubseteq A$, for $A, B \in P$. Hence the relation $\subseteq$ is antisymmetric.
- If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$, for all $A, B, C \in P$. Hence the relation $\subseteq$ is transitive.

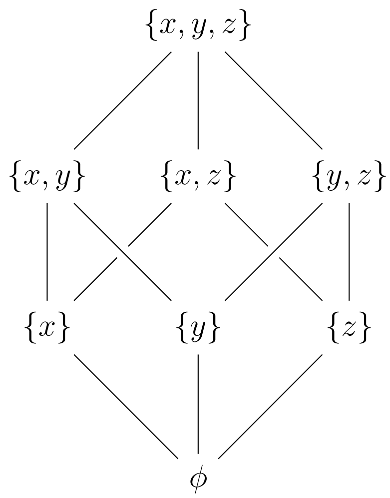
Example 2.1.5. Consider set of all real-valued functions on the interval $[0,1]$. Define $f \leq g$ if $f(t) \leq g(t)$ for all $t \in [0,1]$. The partially ordered set is $P = \{f, g, h, i\}$ where $f(x) = 2x, g(x) = e^x, h(x) = \dfrac{1}{x}, i(x) = \dfrac{1}{x^2}$ where $x \neq 0$. To verify this is the poset

- For $f \in P$ and $t \in [0,1]$, then $f(t) \leq f(t)$. Similarly for $g, h, i \in P$. Hence the relation $\leq$ is reflexive.
- If $f(t) \leq g(t)$ and $g(t) \leq f(t)$ then $f(t) = g(t)$, for all $f, g \in P$ and $t \in [0,1]$. Hence the relation $\leq$ is antisymmetric.
- If $f(t) \leq g(t)$ and $g(t) \leq h(t)$ then $f(t) \leq h(t)$, for all $f, g, h \in P$ and $t \in [0,1]$. Hence the relation $\leq$ is transitive.

Example 2.1.6. The set of normal subgroups of some group with a binary $\triangleleft$. Consider $G = S_4$ be the symmetric group and $P = \{\{1\}, V_4, A_4, S_4\}$ is the set of normal subgroups of S_4. Then $(P, \trianglelefteq)$ is the poset. To check that

- For $V_4 \in P$ then $V_4 \triangleleft V_4$. Hence the relation $\triangleleft$ is reflexive.
- The relation $\triangleleft$ is antisymmetric, since if $V_4 \triangleleft A_4$ then $A_4 \not\triangleleft V_4$ for $V_4, A_4 \in P$.
- The relation $\triangleleft$ is transitive, since if $V_4 \triangleleft A_4$ and $A_4 \triangleleft S_4$ then $V_4 \triangleleft S_4$, for $V_4, A_4, S_4 \in P$.

Example 2.1.7. The most popular example of partially ordered set is Hasse diagram. Hasse diagrams were introduced by a German mathematician Helmut Hasse. He showed a three-element set $\{x, y, z\}$ whose poset diagram is shown below:



Definition 2.1.8. Let $(P, \leq)$ be a partially ordered set. For any $x, y \in P$ the interval from x to y is $[x, y] = \{z \in P : x \leq z \leq y\}$. The length of $[x, y]$ is the length of the longest chain in $[x, y]$ and denoted by $l[x, y]$. A poset $(P, \leq)$ is locally finite if every interval of P is finite and it is bounded if there is an integer n such that $l[x, y] \leq n$ for all $[x, y]$ in $(P, \leq)$, otherwise $(P, \leq)$ is unbounded.

Definition 2.1.9. Let $(P, \leq)$ be a locally finite poset. Let F be a field of characteristic p. We define a set

$$I(P, F) = \{f : P \times P \longrightarrow F : f(x, y) = 0 \quad \text{if} \quad x \nleq y\}.$$

Now we can define the addition operation on $I(P, F)$ as follows:

$$(f + g)(x, y) = f(x, y) + g(x, y);$$

for all $f, g \in I(P, F)$ and $x, y \in P$.
We define the scalar multiplication on $I(P, F)$ as follows:

$$(\lambda f)(x, y) = \lambda f(x, y);$$

for all $f, g \in I(P, F)$, $x, y \in P$ and $\lambda \in F$.
The set $I(P, F)$ is called the incidence algebra of P over F and its elements are called incidence functions. Ii is clear that $I(P, F) \neq \phi$.

In fact, we want to define a third operation on $I(P, F)$. This operation is called convolution product. Consider the following definition:

Definition 2.1.10. Let $(P, \leq)$ be a locally finite poset. Let $I(P, F)$ be the incidence algebra of P over F. Take any $f, g \in I(P, F)$. The convolution product of f and g can be defined as:

$$(f * g)(x, y) = \begin{cases} \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y), & \text{if} \quad x \leq y; \\ 0_F, & \text{Otherwise.} \end{cases}$$

Remarks. Now let us ask the following questions:

- Is the convolution product commutative?
- Is the convolution product associative?
- Does it distributive over addition operation?

The answer of these questions are: this operation is not commutative unless any two elements of P are incomparable. Obviously $(f * g)(x, y) \neq (g * f)(x, y)$ if $x \leq y$.
Let us just verify the distributive law as follows:
For $f, g, h \in I(P, F)$ we have

$$\begin{aligned}
\Big(f * (g + h)\Big)(x, y) &= \sum_{x \leq z \leq y} f(x, z) \cdot \big(g + h\big)(z, y) \\
&= \sum_{x \leq z \leq y} f(x, z)\Big(g(z, y) + h(z, y)\Big) \\
&= \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y) + \sum_{x \leq z \leq y} f(x, z) h(z, y) \\
&= \big(f * g\big)(x, y) + \big(f * h\big)(x, y).
\end{aligned}$$

In the following lemma we show the convolution product is an associative.

Proposition 2.1.11. The incidence algebra $I(P, F)$ is an associative F-algebra with identity.

Proof. For $f, g, h \in I(P, F)$ we have

$$\begin{aligned}
\Big(f * (g * h)\Big)(x, y) &= \sum_{x \leq z \leq y} f(x, z)\big(g * h\big)(z, y) \\
&= \sum_{x \leq z \leq y} f(x, z)\Big(\sum_{z \leq w \leq y} g(z, w) h(w, y)\Big) \\
&= \sum_{x \leq w \leq y} \Big(\sum_{x \leq z \leq w} f(x, z) g(z, w)\Big) h(w, y) \\
&= \sum_{x \leq w \leq y} \big(f * g\big)(x, w) h(w, y) \\
&= \Big((f * g) * h\Big)(x, y).
\end{aligned}$$

□

Let us define incidence functions of the incidence algebra $I(P, F)$.

Definition 2.1.12. Let $(P, \leq)$ be a locally finite poset. Let $I(P, F)$ be the incidence algebra of P over F. Then we say that

(a) The zero map $0(x, y) = 0$, which is in every incidence algebras.

(b) The kronecker delta function of $I(P,F)$ is defined by

$$\delta(x,y) := \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise.} \end{cases}$$

This function simply detects when two elements are the same. More than that it is considered two sided identity of the incidence algebra under convolution $f * \delta = \delta * f = f$, for any $f \in I(P,F)$.

(c) The zeta function of $I(P,F)$ is defined by

$$\zeta(x,y) := \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

(d) The lambda function of $I(P,F)$ is defined by

$$\lambda(x,y) := \begin{cases} 1 & \text{if} \quad x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

(e) The chain function of $I(P,F)$.

$$\eta := \zeta - \delta.$$

(f) The cover function of $I(P,F)$.

$$\kappa := \lambda - \delta.$$

(g) The möbius function of $I(P,F)$.

$$\mu := \zeta^{-1}.$$

We will explain the definition of the möbius function in detail later.

(h) The length function of $I(P,F)$.

$$\rho(x,y) := l[x,y].$$

Remark. We want to ask: for any $f \in I(P,F)$, is there a function $f^{-1} \in I(P,F)$ such that $f^{-1} * f = \delta$? we mean that, is the function f invertible?

The answer is no. Because there is the zero function has no inverse as $0*f = 0$ for any $f \in I(P, F)$. In particular there is no f such that $0*f = \delta$. In general, for any $x \in P$ there is a function $f \in I(P, F)$ such that $f(x, x) = 0$, we note that f has no inverse as follows:
For any $g \in I(P, F)$

$$(f * g)(x, x) = \sum_{x \leq z \leq x} f(x, z)g(z.x) = f(x, x)g(x, x) = 0 \neq \delta(x, x) = 1.$$

So, it is impossible for f to have an inverse if $f(x, x) = 0$.
We conclude that f has an inverse in case $f(x, x) \neq 0$ for all $x \in P$. In fact f is invertible. We prove this in the following theorem.

Theorem 2.1.13. Suppose that P is a finite poset and $I(P, F)$ is the incidence algebra of P over F. An element $f \in I(P, F)$ is invertible if and only if $f(x, x) \neq 0$ for all $x \in P$.

Proof. If f is an invertible, this mean there is $f^{-1} \in I(P, F)$ such that $f * f^{-1} = f^{-1} * f = \delta$. Then for all $x \in P$

$$\delta(x, x) = 1 = (f * f^{-1})(x, x) = f(x, x)f^{-1}(x, x).$$

Thus $f(x, x) \neq 0$.
Conversely, for all $x \in P$ suppose that $f(x, x) \neq 0$. We construct f^{-1} inductively.
First we will define it on all of the pairs (x, x) with $x \in P$, and after that we will extend it to pairs (x, y) with $x < y \in P$.
For any $x \in P$, we define $f^{-1}(x, x) = \dfrac{1}{f(x, x)}$. Then by definition we have

$$(f * f^{-1})(x, x) = \sum_{x \leq z \leq x} f(x, z)f^{-1}(z, x) = f(x, x)\frac{1}{f(x, x)} = 1 = \delta(x, x).$$

Similarly $(f^{-1} * f)(x, x) = 1 = \delta(x, x)$. Thus f is an invertible element in $I(P, F)$.
For any $x < y \in P$, we define the left inverse as follows:

$$f^{-1}(x, y) = -\frac{1}{f(y, y)}\Big(\sum_{x \leq z < y} f^{-1}(x, z)f(z, y)\Big).$$

We want to verify that we have made an inverse. It must be $(f^{-1}*f)(x,y) = 0$ for $x < y$ then we have

$$\begin{aligned}
(f^{-1}*f)(x,y) &= \sum_{x\leq z\leq y} f^{-1}(x,z)f(z,y) \\
&= \Big(\sum_{x\leq z<y} f^{-1}(x,z)f(z,y)\Big) + f^{-1}(x,y)f(y,y) \\
&= \Big(\sum_{x\leq z<y} f^{-1}(x,z)f(z,y)\Big) - \frac{1}{f(y,y)}\Big(\sum_{x\leq z<y} f^{-1}(x,z)f(z,y)\Big)f(y,y) \\
&= \Big(\sum_{x\leq z<y} f^{-1}(x,z)f(z,y)\Big) - \Big(\sum_{x\leq z<y} f^{-1}(x,z)f(z,y)\Big) \\
&= 0 = \delta(x,y).
\end{aligned}$$

The right inverse we can define it as follows

$$f^{-1}(x,y) = -\frac{1}{f(x,x)}\Big(\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big).$$

We check that $(f*f^{-1})(x,y) = 0$ then we have

$$\begin{aligned}
(f*f^{-1})(x,y) &= \sum_{x\leq z\leq y} f(x,z)f^{-1}(z,y) \\
&= f(x,x)f^{-1}(x,y) + \Big(\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big) \\
&= f(x,x)\frac{1}{f(x,x)}\Big(-\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big) + \Big(\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big) \\
&= \Big(-\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big) + \Big(\sum_{x<z\leq y} f(x,z)f^{-1}(z,y)\Big) \\
&= 0 = \delta(x,y).
\end{aligned}$$

We get that f has a left inverse and a right inverse. Thus f is an invertible element in the incidence algebra $I(P,F)$. □

Theorem 2.1.14. Let P be a locally finite poset. Let $I(P,F)$ be the incidence algebra of P over F. Then $J_x(P) = \{f \in I(P,F) : f(x,x) = 0\}$ is a maximal two-sided ideal in $I(P,F)$, for all $x \in P$.

Proof. First to prove that $J_x(P)$ is an ideal of $I(P,F)$. Let $f \in J_x(P)$ and $g \in I(P,F)$ we have

$$\begin{aligned}
(g*f)(x,x) &= g(x,x)f(x,x) = g(x,x)0 = 0 \\
(f*g)(x,x) &= f(x,x)g(x,x) = 0g(x,x) = 0.
\end{aligned}$$

So, $g * f, f * g \in J_x(P)$. Therefore $J_x(P)$ is an ideal of $I(P, F)$.
Secondly to prove that $J_x(P)$ is a maximal ideal of $I(P, F)$. Suppose that I is an ideal of $I(P, F)$ such that $J_x(P) < I \leq I(P, F)$ then there is $g \in I(P, F)$ with $g \in I$ and $g \notin J_x(P)$, hence $g(x, x) \neq 0$. But by Theorem 2.1.13 g is unit, but since $g \in I$ then $I = I(P, F)$. So, $J_x(P)$ is a maximal ideal in $I(P, F)$. □

Corollary 2.1.15. Let P be a locally finite poset. Let $I(P, F)$ be the incidence algebra of P over F. Then $I(P, F)/J_x(P) \cong F$.

Theorem 2.1.16. (Stanley)[4, 13] Let P and Q be locally finite posets. Let $I(P, F)$ and $I(Q, F)$ be two incidence algebras over F. Then

$$I(P, F) \cong I(Q, F) \Rightarrow P \cong Q.$$

In other words If $I(P, F)$ and $I(Q, F)$ are isomorphic as F-algebra then P and Q are isomorphic as posets.

Proof. We shall show how the order set P can be uniquely recovered from the ring $I(P, F)$
Firstly, we will define the element $e_x, e_{x,y}$ for each $x, y \in P$ with

$$e_x(u, v) = \begin{cases} 1, & \text{if} \quad u = v = x; \\ 0, & \text{Otherwise.} \end{cases}$$

$$e_{x,y}(u, v) = \begin{cases} 1, & \text{if} \quad x = u, y = v; \\ 0, & \text{Otherwise.} \end{cases}$$

Note that $\{e_x : x \in P\}$ is a system of orthogonal idempotent in $I(P, F)$. Since

- $e_x \in I(P, F)$ for each $x \in P$ because for all
  $u \nleq v \Rightarrow u \neq v \Rightarrow e_x(u, v) = 0$
- $e_x * e_x = e_x$ hence e_x is an idempotent
- $e_x * e_y = e_y * e_x = 0$ if $x \neq y$ hence e_x, e_y are orthogonal.

Secondly, since $e_x * e_x = e_x$ then the element e_x ia an idempotent whose image in $I(P, F)/J_x(P)$ is the identity element in F.
Thirdly, we define an order relation P' on the e_x as follows $e_x \leq_{P'} e_y$ if and only if $e_x * I(P, F) * e_y \neq \{0\}$.
We want to prove that $P' \cong P$, it enough to show that $e_x \leq_{P'} e_y \Leftrightarrow x \leq_P y$,

to show that $e_x * I(P,F) * e_y \neq \{0\} \Leftrightarrow x \leq_P y$.
Now let $f \in I(P,F)$ then $e_x * f * e_y \in e_x * I(P,F) * e_y$ and note that

$$e_x * f * e_y = f(x,y)e_{x,y}. \tag{2.1}$$

Since

$$\begin{aligned}
\big(e_x * f * e_y\big)(u,v) &= \Big(\big(e_x * f\big)(u,v)\Big) * e_y(u,v) \\
&= \left(\begin{cases} 0, & \text{if} \quad u \neq x; \\ f(x,v), & \text{if} \quad u = x. \end{cases} \right) * e_y(u,v) \\
&= \begin{cases} 0, & \text{if} \quad u \neq x, v \neq y; \\ f(x,y), & \text{if} \quad u = x, v = y. \end{cases} \\
&= f(x,y) \begin{cases} 0, & \text{if} \quad u \neq x, v \neq y; \\ 1, & \text{if} \quad u = x, v = y. \end{cases} \\
&= f(x,y)e_{x,y}(u,v).
\end{aligned}$$

Now suppose that

$$\begin{aligned}
e_x \leq_{P'} e_y &\Longleftrightarrow e_x * I(X,P) * e_y \neq \{0\} \\
&\Longleftrightarrow e_x * f * e_y \neq 0 \qquad f \in I(X,P) \\
&\Longleftrightarrow f(x,y)e_{x,y} \neq 0 \qquad \text{from}(2.1) \\
&\Longleftrightarrow f(x,y) \neq 0 \\
&\Longleftrightarrow x \leq_P y.
\end{aligned}$$

So, for each $e_x \leq_{P'} e_y \Leftrightarrow x \leq_P y$. Hence

$$P \cong P'. \tag{2.2}$$

Fourthly, in the same way since $e_x, e_y \in I(Q,F)$ also.
Then $e_x * I(Q,F) * e_y \neq \{0\} \Leftrightarrow x \leq_Q y$.
But since $I(P,F) \cong I(Q,F)$ then
$e_x * I(Q,F) * e_y \neq \{0\} \Leftrightarrow e_x * I(P,F) * e_y \neq \{0\}$.
So,

$$e_x * I(P,F) * e_y \neq \{0\} \Leftrightarrow x \leq_Q y$$

$$e_x \leq_{P'} e_y \Leftrightarrow x \leq_Q y.$$

This mean

$$P' \cong Q. \tag{2.3}$$

Hence by (2.2) and (2.3) $P \cong Q$.
Now, the proof will be complete if we show that given any orthogonal idempotent set such that $\{f_x, x \in P\}$ then it isomorphic to $\{e_x, x \in P\}$, it enough to find isomorphism ϕ of $I(P, F)$ where $\phi(e_x) = f_x$ for all $x \in P$. Define the map

$$\phi : I(P, F) \longrightarrow I(P, F)$$

with $\phi(g) = hgh^{-1}$ where $h = \sum_{t \in P} f_t e_t$, ϕ is inner automorphism, just we prove that $\phi(e_x) = f_x$. Now

$$\begin{aligned} he_x &= \big(\sum_{t \in P} f_t e_t\big) e_x \\ &= \sum_{t \in P} f_t e_t e_x \\ &= f_x e_x e_x. \end{aligned}$$

So, we get

$$he_x = f_x e_x. \tag{2.4}$$

Also

$$\begin{aligned} f_x h &= f_x\big(\sum_{t \in P} f_t e_t\big) \\ &= \sum_{t \in P} f_x f_t e_t \\ &= f_x f_x e_x. \end{aligned}$$

We get

$$f_x h = f_x e_x. \tag{2.5}$$

Hence by (2.4) and (2.5)

$$\begin{aligned} he_x &= f_x h \\ he_x h^{-1} &= f_x \\ \phi(e_x) &= f_x. \end{aligned}$$

□

Definition 2.1.17. Take the zeta function ζ . The möbius function μ is an inverse to the zeta function. We can present a new definition of the möbius function as follows:

$$\mu(x, x) = \frac{1}{\zeta(x, x)} = 1$$

$$\mu(x,y) = -\sum_{x\leq z<y} \mu(x,z) = -\sum_{x<z\leq y} \mu(z,y) \quad \text{if} \quad x<y,$$

for any $x, y, z \in P$ and if $x \not\leq y$ implies $\mu(x,y) = 0$.

Theorem 2.1.18. (Möbius Inversion Formula)
Take any poset P. Suppose that P has unique minimal element m. Let ϕ and ψ be functions from P to F. If ϕ satisfies

$$\phi(x) = \sum_{y\leq x} \psi(y) \quad \text{for all} \quad x \in P.$$

Then

$$\psi(x) = \sum_{y\leq x} \phi(y)\mu(y,x) \quad \text{for all} \quad x \in P.$$

Proof. Suppose that $\psi : P \longrightarrow F$ and defined $\phi : P \longrightarrow F$ as follows:

$$\phi(x) = \sum_{y\leq x} \psi(y) \quad \text{for all} \quad x \in P.$$

Let m be an unique minimal element of P. In other words $m \leq y$ for all $y \in P$. Now we define the functions $f, g \in I(P, F)$ as follows:

- $f(m, x) = \psi(x)$ for all x, and $f(y, z) = 0$ for all other undefined values.
- $g(m, x) = \phi(x)$ for all x, and $g(y, z) = 0$ for all other undefined values.

Now we have for any $x \in P$

$$\begin{aligned} g(m,x) &= \phi(x) \\ &= \sum_{y\leq x} \psi(y) \\ &= \sum_{m\leq y\leq x} f(m,y) \\ &= \sum_{m\leq y\leq x} f(m,y)\zeta(y,x) \\ &= (f*\zeta)(m,x). \end{aligned}$$

Now for any $y, z \in P$ with $y \neq m$ and $g(y, z) = 0$ we have

$$(f*\zeta)(y,z) = \sum_{y\leq t\leq z} f(y,t)\cdot\zeta(t,z) = \sum_{y\leq t\leq z} 0\cdot\zeta(t,z) = 0.$$

So, we have proved that $g = f * \zeta$. Also we know that $g * \mu = f$. Now we prove the required

$$\begin{aligned}\psi(x) &= f(m,x)\\ &= (g*\mu)(m,x)\\ &= \sum_{m\le y\le x} g(m,y)\mu(y,x)\\ &= \sum_{y\le x}\phi(y)\mu(y,x).\end{aligned}$$

It is required to prove it. □

Corollary 2.1.19. Given any poset P. Suppose that P has unique maximal element n. Let s and r be functions from P to F. If s satisfies

$$s(x) = \sum_{y\ge x} r(y) \quad \text{for all} \quad x \in P.$$

Then

$$r(x) = \sum_{y\ge x}\mu(x,y)s(y) \quad \text{for all} \quad x \in P.$$

Proof. Suppose that $r : P \longrightarrow F$ and defined $s : P \longrightarrow F$ as follows:

$$s(x) = \sum_{y\ge x} r(y) \quad \text{for all} \quad x \in P.$$

Let n be an unique maximal element of P. This mean $m \ge y$ for all $y \in P$. Now we define the functions $h, l \in I(P, F)$ as follows:

- $h(x,m) = r(x)$ for all x, and $h(y,z) = 0$ for all other undefined values.
- $l(x,m) = s(x)$ for all x, and $l(y,z) = 0$ for all other undefined values.

Now we have for any $x \in P$

$$\begin{aligned}l(x,m) &= s(x)\\ &= \sum_{y\ge x} r(y)\\ &= \sum_{m\ge y\ge x} h(y,m)\\ &= \sum_{m\ge y\ge x}\zeta(x,y)h(y,m)\\ &= (\zeta*h)(x,m).\end{aligned}$$

Now for any $y, z \in P$ with $y \neq m$ and $l(y,z) = 0$ we have

$$(\zeta * h)(y,z) = \sum_{z \geq t \geq y} \zeta(y,t) \cdot h(t,z) = \sum_{z \geq t \geq y} \zeta(y,t) \cdot 0 = 0.$$

So, we have proved that $l = \zeta * h$. Also we know that $\mu * l = h$. Now we prove the required

$$\begin{aligned} r(x) &= h(x,m) \\ &= (\mu * l)(x,m) \\ &= \sum_{m \geq y \geq x} \mu(x,y) l(y,m) \\ &= \sum_{y \geq x} \mu(x,y) s(y). \end{aligned}$$

It is required to prove it. □

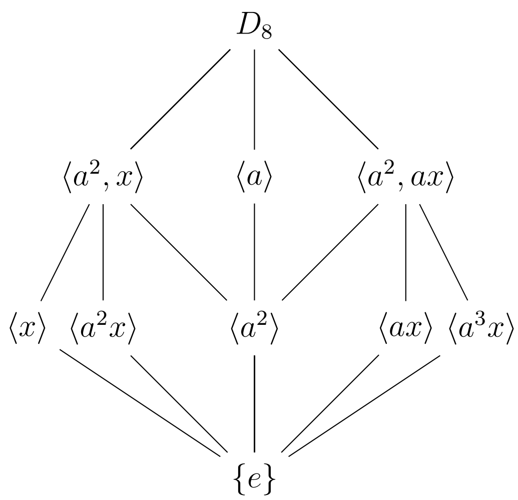
2.2 Modular incidence algebras

The incidence algebra was defined over a commutative ring. This ring can be a field. This field has a characteristic zero or has characteristic $p > 0$, where p is a prime number. Some authors studied an incidence algebra over a field which has a characteristic zero. In this case, the study is called ordinary.
Now, we will study incidence algebra over a field which has characteristic $p > 0$. For the prime number p, either divides the order of G or not divides it. We will consider the following two questions, is there a difference between the two cases and what kind of representation in each case?. This is what we study in this section.

Example 2.2.1. We have the dihedral group D_8 is 2-group of order 8

$$D_8 = \langle a, x : a^4 = e, x^2 = e, (ax)^2 = e\rangle$$
$$D_8 = \{e, a, a^2, a^3, x, ax, a^2x, a^3x\}.$$

The subgroups lattice of D_8 is



We take $P_1 = \big\{\{e\}, \langle x\rangle, \langle a^2, x\rangle, D_8\big\}$ to be a locally finite poset, ordered by inclusion where $H_1 \leq H_2$ if $H_1 \subseteq H_2, H_1, H_2 \in P_1$. Consider $F = \mathbb{Z}_2$ is the field which has characteristic two ($p = 2$). The incidence algebra of P_1 over $\mathbb{Z}_2$ is

$$I(P_1, \mathbb{Z}_2) = \{f : P_1 \times P_1 \longrightarrow \mathbb{Z}_2 : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

In this case we can write the möbius function as the following

$$\mu_G(H_1, H_2) = \begin{cases} (-1)^k p^{\binom{k}{2}}, & \text{if} \quad H_1 \text{is a normal subgroup of } H_2 \quad \text{and } H_2/H_1 \cong (\mathbb{Z}/p\mathbb{Z})^k; \\ & \text{where} \quad k \in \mathbb{N}. \\ 0, & \text{Otherwise.} \end{cases}$$

We will calculate the möbius function

- $\{e\} \lhd \langle x \rangle$ and $\langle x \rangle / \{e\} \cong \mathbb{Z}/2\mathbb{Z}$,
  $\mu(\{e\}, \langle x \rangle) = (-1)^1 2^{1(1-1)/2} = 1$.
- $\{e\} \lhd \langle a^2, x \rangle$ and $\langle a^2, x \rangle / \{e\} \cong (\mathbb{Z}/2\mathbb{Z})^2$,
  $\mu(\{e\}, \langle a^2, x \rangle) = (-1)^2 2^{\binom{2}{2}} = 0$.
- $\{e\} \lhd D_8$ and $D_8/\{e\} \cong (\mathbb{Z}/2\mathbb{Z})^3$,
  $\mu(\{e\}, D_8) = (-1)^3 2^{\binom{3}{2}} = (-1)2^3 = 0$.
- $\langle x \rangle \lhd \langle a^2, x \rangle$ and $\langle a^2, x \rangle / \langle x \rangle \cong \mathbb{Z}/2\mathbb{Z}$,
  $\mu(\langle x \rangle, \langle a^2, x \rangle) = 1$.
- $\langle x \rangle$ is not normal in D_8, so $\mu(\langle x \rangle, D_8) = 0$.
- $\langle a^2, x \rangle \lhd D_8$ and $D_8 / \langle a^2, x \rangle \cong \mathbb{Z}/2\mathbb{Z}$,
  $\mu(\langle a^2, x \rangle, D_8) = 1$.

Now, we study the same example with change the field to be $F = \mathbb{Z}_3$ which has characteristic three ($p = 3$). The incidence algebra of P_1 over $\mathbb{Z}_3$ is

$$I(P_1, \mathbb{Z}_3) = \{f : P_1 \times P_1 \longrightarrow \mathbb{Z}_3 : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

So, the möbius function in this case is

- $\mu(\{e\}, \langle x \rangle) = 2$
- $\mu(\{e\}, \langle a^2, x \rangle) = 2$
- $\mu(\{e\}, D_8) = 1$
- $\mu(\langle x \rangle, \langle a^2, x \rangle) = 2$
- $\mu(\langle x \rangle, D_8) = 0$
- $\mu(\langle a^2, x \rangle, D_8) = 2$.

We compare the result in both cases $F = \mathbb{Z}_2$ and $F = \mathbb{Z}_3$ in the following table.

The möbius function	$F = \mathbb{Z}_2$	$F = \mathbb{Z}_3$
$\mu(\{e\}, \langle x \rangle)$	1	2
$\mu(\{e\}, \langle a^2, x \rangle)$	0	2
$\mu(\{e\}, D_8)$	0	1
$\mu(\langle x \rangle, \langle a^2, x \rangle)$	1	2
$\mu(\langle x \rangle, D_8)$	0	0
$\mu(\langle a^2, x \rangle, D_8)$	1	2

In case $F = \mathbb{Z}_2$ we note that

- If H_1 is a maximal normal subgroup of H_2 then $\mu(H_1, H_2) = 1$
- Otherwise the möbius function is equal to zero.

In case $F = \mathbb{Z}_3$ we note that

- $\mu(\{e\}, D_8) = 1$
- If H_1 is a normal subgroup of H_2 then $\mu(H_1, H_2) = 2$
- Otherwise the möbius function is equal to zero.

The representation in the first case we call a modular incidence algebra and the second case we call an ordinary incidence algebra since the characteristic 3 of $\mathbb{Z}_3$ does not divide the order $|G|$.
Now we take another a locally finite poset, let $P_2 = \big\{\{e\}, \langle a^2\rangle, \langle a\rangle, D_8\big\}$ ordered by inclusion. If $F = \mathbb{Z}_2$. The incidence algebra

$$I(P_2, \mathbb{Z}_2) = \{f : P_2 \times P_2 \longrightarrow \mathbb{Z}_2 : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

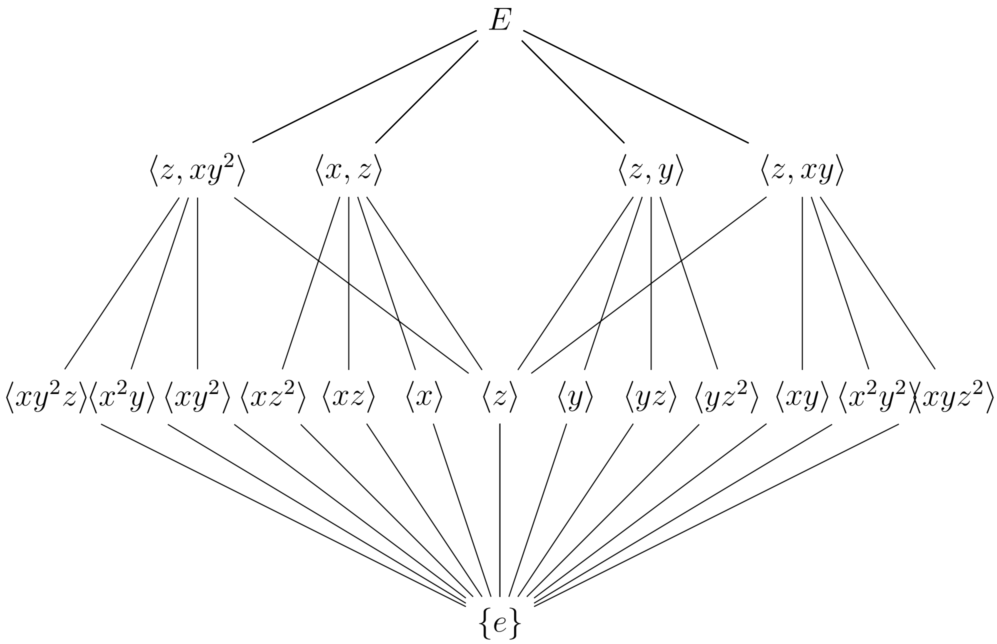
We will calculate the möbius function

- $\mu(\{e\}, \langle a^2\rangle) = 1$
- $\mu(\{e\}, \langle a\rangle) = 0$
- $\mu(\{e\}, D_8) = 0$
- $\mu(\langle a^2\rangle, \langle a\rangle) = 1$
- $\mu(\langle a^2\rangle, D_8) = 0$
- $\mu(\langle a\rangle, D_8) = 1.$

Example 2.2.2. The extra special 3-group has the presentation:

$$E = \langle x, y, z| \quad x^3 = y^3 = z^3 = e, [x, y] = z, \quad [x, z] = [z, y] = e\rangle$$
$$E = \{e, x, x^2, y, y^2, z, z^2, xy, xy^2, xz, xz^2, x^2y, x^2y^2, x^2z, x^2z^2, yz, yz^2, y^2z, y^2z^2, xyz, xyz^2, xy^2z, xy^2z^2, x^2yz, x^2yz^2, x^2y^2z, x^2y^2z^2\}.$$

The order of the group E is equal to 27. The group E has 13 subgroups of order 3 and 4 maximal subgroups of order 9. The subgroups lattice of E is



We take $P = \big\{\{e\}, \langle x\rangle, \langle x, z\rangle, E\big\}$ to be a locally finite poset, ordered by inclusion. Take a field of characteristic 2, let $F = \mathbb{Z}_2$. The incidence algebra of P over $\mathbb{Z}_2$ is

$$I(P, \mathbb{Z}_2) = \{f : P \times P \longrightarrow \mathbb{Z}_2 : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

We can write the möbius function as the following

$$\mu_G(H_1, H_2) = \begin{cases} (-1)^k p^{\binom{k}{2}}, & \text{if} \quad H_1 \text{is a normal subgroup of } H_2 \quad \text{and } H_2/H_1 \cong (\mathbb{Z}/p\mathbb{Z})^k; \\ & \text{where} \quad k \in \mathbb{N}. \\ 0, & \text{Otherwise.} \end{cases}$$

We will calculate the möbius function

- $\{e\} \lhd \langle x\rangle$ and $\langle x\rangle/\{e\} \cong \mathbb{Z}/3\mathbb{Z}$,
  $\mu(\{e\}, \langle x\rangle) = (-1)^1 3^{1(1-1)/2} = 1.$
- $\{e\} \lhd \langle x, z\rangle$ and $\langle x, z\rangle/\{e\} \cong (\mathbb{Z}/3\mathbb{Z})^2$,
  $\mu(\{e\}, \langle x, z\rangle) = (-1)^2 3^{\binom{2}{2}} = 1.$
- $\{e\} \lhd E$ and $E/\{e\} \cong (\mathbb{Z}/3\mathbb{Z})^3$,
  $\mu(\{e\}, E) = (-1)^3 3^{\binom{3}{2}} = (-1)3^3 = 1.$
- $\langle x\rangle \lhd \langle x, z\rangle$ and $\langle x, z\rangle/\langle x\rangle \cong \mathbb{Z}/3\mathbb{Z}$,
  $\mu(\langle x\rangle, \langle x, z\rangle) = 1.$

- $\langle x\rangle$ is not normal in E, so $\mu(\langle x\rangle, E) = 0$.
- $\langle x, z\rangle \lhd E$ and $E/\langle x, z\rangle \cong \mathbb{Z}/3\mathbb{Z}$.
  $\mu(\langle x, z\rangle, E) = 1$.

Now, we study the same example with change the field to be $F = \mathbb{Z}_3$. The incidence algebra of P over $\mathbb{Z}_3$ is

$$I(P, \mathbb{Z}_3) = \{f : P \times P \longrightarrow \mathbb{Z}_3 : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

So, the möbius function in this case is

- $\mu(\{e\}, \langle x\rangle) = 2$
- $\mu(\{e\}, \langle x, z\rangle) = 0$
- $\mu(\{e\}, E) = 0$
- $\mu(\langle x\rangle, \langle x, z\rangle) = 2$
- $\mu(\langle x\rangle, E) = 0$
- $\mu(\langle x, z\rangle, E) = 2$.

We compare the result in both cases $F = \mathbb{Z}_2$ and $F = \mathbb{Z}_3$ in the following table.

The möbius function	$F = \mathbb{Z}_2$	$F = \mathbb{Z}_3$
$\mu(\{e\}, \langle x\rangle)$	1	2
$\mu(\{e\}, \langle x, z\rangle)$	1	0
$\mu(\{e\}, E)$	1	0
$\mu(\langle x\rangle, \langle x, z\rangle)$	1	2
$\mu(\langle x\rangle, E)$	0	0
$\mu(\langle x, z\rangle, E)$	1	2

In case $F = \mathbb{Z}_2$ we note that

- $\mu(\{e\}, E) = 1$
- If H_1 is a normal subgroup of H_2 then $\mu(H_1, H_2) = 1$
- Otherwise the möbius function is equal to zero.

In case $F = \mathbb{Z}_3$ we note that

- If H_1 is a maximal normal subgroup of H_2 then $\mu(H_1, H_2) = 2$

- Otherwise the möbius function is equal to zero.

The representation in the first case is an ordinary incidence algebra and in the second case, the representation is a modular incidence algebra.

Remark. We deduce from Example 2.2.1 and Example 2.2.2 that.
If p is a prime number and $F = \mathbb{Z}_p$. If G is a finite p-group and P is a set of subgroups of G ordered by inclusion. The incidence algebra is

$$I(P, \mathbb{Z}_p) = \{f : P \times P \longrightarrow \mathbb{Z}_p : f(H_1, H_2) = 0 \quad \text{if} \quad H_1 \nleq H_2\}.$$

The möbius function is

$$\mu_G(H_1, H_2) = \begin{cases} -1, & \text{if} \quad H_1 \text{is a maximal normal subgroup of } H_2; \\ 0_F, & \text{Otherwise.} \end{cases}$$

Here the type of representation is modular representation of incidence algebra.

Example 2.2.3. Take $M = \{2, 2, 2, 2, 3, 3, 5\}$ to be a multiset. We take a finite sub-multisets of multiset M form a locally finite poset, as follows

$$P = \Big\{\{2, 2, 3\}, \{2, 2, 3, 5\}, \{2, 2, 3, 3, 5\}, \{2, 2, 2, 3, 3, 5\}, \{2, 2, 2, 2, 3, 3, 5\}\Big\}.$$

This poset ordered by inclusion where $S \leq T$ if $S \subseteq T, S, T \in P$. Let $F = \mathbb{Z}_2$ be the field has characteristic 2. The incidence algebra of P over $\mathbb{Z}_2$ as follows

$$I(P, \mathbb{Z}_2) = \{f : P \times P \longrightarrow \mathbb{Z}_2 : f(S, T) = 0 \quad \text{if} \quad S \nleq T\}.$$

We can write the möbius function as the following

$$\mu(S, T) = \begin{cases} 0, & \text{if} \quad T \setminus S \quad \text{is a propper multiset (has repeated elements) ;} \\ (-1)^{|T \setminus S|}, & \text{if} \quad T \setminus S \quad \text{is a set (has no repeated elements).} \end{cases}$$

We will calculate the möbius function over the field $\mathbb{Z}_2$.
$\mu(\{2, 2, 3\}, \{2, 2, 3, 5\}) = 1$
$\mu(\{2, 2, 3\}, \{2, 2, 3, 3, 5\}) = (-1)^2 = 1$
$\mu(\{2, 2, 3\}, \{2, 2, 2, 3, 3, 5\}) = (-1)^3 = 1$
$\mu(\{2, 2, 3\}, \{2, 2, 2, 2, 3, 3, 5\}) = 0$
$\mu(\{2, 2, 3, 5\}, \{2, 2, 3, 3, 5\}) = 1$
$\mu(\{2, 2, 3, 5\}, \{2, 2, 2, 3, 3, 5\}) = 1$
$\mu(\{2, 2, 3, 5\}, \{2, 2, 2, 2, 3, 3, 5\}) = 0$
$\mu(\{2, 2, 3, 3, 5\}, \{2, 2, 2, 3, 3, 5\}) = 1$
$\mu(\{2, 2, 3, 3, 5\}, \{2, 2, 2, 2, 3, 3, 5\}) = 0$
$\mu(\{2, 2, 2, 3, 3, 5\}, \{2, 2, 2, 2, 3, 3, 5\}) = 1.$

2.3 Group action on modular incidence algebras

In this section, we will explain the definition of the group action on a poset and give some examples. We will apply the definition of a group action to a modular incidence algebra. We will explain how to make a modular incidence G-algebra as an interior G-algebra.

Definition 2.3.1. Let G be a finite group. Let P be a poset. The action of the group G on the poset P is a function $G \times P \longrightarrow P$. It is defined by $(g, x) = x^g = gx$ such that $x^e = x$ and $(x^g)^h = x^{(gh)}$ for all $x \in P$ and $g, h \in G$. We say that P is a G-poset.

Example 2.3.2. If $G = \mathbb{Z}$ is the group and P is the set of even numbers of $\mathbb{Z}$. Then G acts on P by left multiplication, we define the function by $(g, x) \longmapsto gx$ for all $x \in P$ and $g, h \in \mathbb{Z}$.

Example 2.3.3. If $D_8 = \langle a, x : a^4 = e, x^2 = e, (ax)^2 = e \rangle$ is dihedral group of order 8 and take $P = \big\{\{e\}, \langle a^2 \rangle, \langle a^2, x \rangle, D_8\big\}$ to be a poset, ordered by inclusion. The group D_8 acts on P by $x^g = gxg^{-1}$ for all $x \in P$ and $g \in G$. To check that this is an action, we see that $\langle a^2 \rangle^e = e\langle a^2 \rangle e^{-1} = \langle a^2 \rangle$ and

$$\begin{aligned}
(\langle a^2 \rangle^a)^x &= x(\langle a^2 \rangle^a)x^{-1} \\
&= x(a\{e, a^2\}a^{-1})x \\
&= x(a\{e, a^2\}a^3)x \\
&= x(\{a, a^3\}a^3)x \\
&= x\{e, a^2\}x \\
&= \{x, a^2x\}x \\
&= \{e, a^2\} \\
&= \langle a^2 \rangle.
\end{aligned}$$

And

$$\begin{aligned}
\langle a^2 \rangle^{(ax)} &= (ax)\langle a^2 \rangle(ax)^{-1} \\
&= (ax)\{e, a^2\}(ax) \\
&= \{ax, a^3x\}(ax) \\
&= \{e, a^2\} \\
&= \langle a^2 \rangle.
\end{aligned}$$

Hence $(\langle a^2 \rangle^a)^x = \langle a^2 \rangle^{(ax)}$.

Remark. In Example 2.3.3, we have more choice to choose P. The first choice is mentioned in the example $P_1 = \big\{\{e\}, \langle a^2\rangle, \langle a^2, x\rangle, D_8\big\}$. The action $x^g = gxg^{-1}$ is well defined. The poset P_1 receives this action. This kind serves us. While the second choice is $P_2 = \big\{\{e\}, \langle x\rangle, \langle a^2, x\rangle, D_8\big\}$. It does not receive and does undefined this action, because

$$\begin{aligned}
(\langle x\rangle^a)^x &= x(\langle x\rangle^a)x^{-1}\\
&= x(a\{e,x\}a^{-1})x\\
&= x(a\{e,x\}a^{3})x\\
&= x(\{a,ax\}a^{3})x\\
&= x\{e,a^2x\}x\\
&= \{x,a^2\}x\\
&= \{e,a^2x\} \notin P_2.
\end{aligned}$$

The following lemma says that the conjugation action preserves the order relation.

Lemma 2.3.4. Let G be a finite group. Let P a poset. If P is a G-poset then $x \le y \Leftrightarrow x^g \le y^g$ where $x, y \in P$ and $g \in G$.

Proof. Suppose that P is a G-poset. Then there is a function $G \times P \longrightarrow P$ which is defined by $(g, x) = x^g = gx$ for all $x \in P$ and $g \in G$. Let $x, y \in P$ then $(g, x) = x^g = gx$ and $(g, y) = y^g = gy$ where $g \in G$. Now if $x \le y$ clearly that $gx \le gy$ so $x^g \le y^g$. Hence

$$x \le y \Rightarrow x^g \le y^g \tag{2.6}$$

Conversely, if $x^g \le y^g$ for all $x, y \in P$ and $g \in G$ then

$$\begin{aligned}
x^g &\le y^g\\
gx &\le gy\\
g^{-1}gx &\le g^{-1}gy\\
x &\le y.
\end{aligned}$$

Hence

$$x^g \le y^g \Rightarrow x \le y \tag{2.7}$$

From (2.6) and (2.7) we get

$$x \le y \Leftrightarrow x^g \le y^g.$$

□

In the following definition we define the action of a finite group on a modular incidence algebra.

Definition 2.3.5. Let G be a finite group. Let P be a locally finite poset which is a G-poset. The modular incidence algebra $I(P,F)$ of P over F can be made in a G-algebra structure by $f^a(x,y) = f(x^a, y^a)$ where $a \in G, x, y \in P$ and $f \in I(P,F)$. We will check the conditions listed in Definition 1.2.14 as follows:

$$\begin{aligned}
f^e(x,y) &= f(x^e, y^e) = f(x,y). \\
(f^b(x,y))^a &= \big(f(x^b, y^b)\big)^a \\
&= f\big((x^b)^a, (y^b)^a\big) \\
&= f\big(x^{ba}, y^{ba}\big) \\
&= f^{ba}(x,y). \\
\big((f+g)^a(x,y)\big) &= (f+g)(x^a, y^a) \\
&= f(x^a, y^a) + g(x^a, y^a) \\
&= f^a(x,y) + g^a(x,y). \\
\big((f*g)^a(x,y)\big) &= (f*g)(x^a, y^a) \\
&= \sum_{x \le z \le y} f(x^a, z^a) \cdot g(z^a, y^a) \\
&= \sum_{x \le z \le y} f^a(x,z) \cdot g^a(z,y). \\
\big(\alpha f\big)^a(x,y) &= \big(\alpha f\big)(x^a, y^a) \\
&= \alpha f(x^a, y^a) \\
&= \alpha f^a(x,y).
\end{aligned}$$

Lemma 2.3.6. If P is a G-poset then $I(P,F)$ is a G-algebra over F.

Proof. If G is a finite group. Suppose that P is a G-poset. From Lemma 2.3.4, for any $x, y \in P$ and $a \in G$, we have $x \le y \Leftrightarrow x^a \le y^a$. For any $f(x,y) \in I(P,F)$ there is $f(x^a, y^a) \in I(P,F)$. By definition $f^a(x,y) = f(x^a, y^a)$. Hence $f^a \in I(P,F)$. We get $I(P.F)$ is a G-algebra over F. □

Example 2.3.7. If $G = E$ is the extra special 3-group.
Take $P = \big\{\{e\}, \langle z\rangle, \langle x, z\rangle, E\big\}$. Let $(P, \subseteq)$ be a locally finite poset. The group E acts on P by $x^g = gxg^{-1}$ for all $x \in P$ and $g \in E$. Let $y \in E$,

$$\langle z\rangle^y = y\langle z\rangle y^{-1} = \langle z\rangle$$

$$\langle x, z\rangle^y = y\langle x, z\rangle y^{-1} = \langle x, z\rangle.$$

Where $\langle z\rangle \subseteq \langle x, z\rangle$ ordered by inclusion then so is $\langle z\rangle^y \subseteq \langle x, z\rangle^y$.
Furthermore, the group E acts on the modular incidence algebra $I(P, \mathbb{Z}_2)$ of P over $\mathbb{Z}_2$.

In general by Definition 2.3.5. Take G to be a finite group and H to be a subgroup of G. Let P a locally finite poset which is a G-poset. Let $A = I(P, F)$ be the modular incidence algebra which is a G-algebra over F. The set of H-fixed points of A is

$$A^H = \{f \in A, f^h(x, y) = f(x, y), \text{for all} \quad h \in H, x, y \in P\}$$
$$A^H = \{f \in A, f(x^h, y^h) = f(x, y), \text{for all} \quad h \in H, x, y \in P\}.$$

The set of G-fixed points of A is

$$A^G = \{f \in A, f^g(x, y) = f(x, y), \text{for all} \quad g \in G, x, y \in P\}$$
$$A^G = \{f \in A, f(x^g, y^g) = f(x, y), \text{for all} \quad g \in G, x, y \in P\}.$$

Then the relative trace map

$$t_H^G : A^H \longrightarrow A^G, \quad t_H^G(f) = \sum_{t \in T} f^t,$$

for all $f \in A^H$. Where T to be a co-set representative of H in G.
We know that $A_H^G = t_H^G(A^H)$ is an ideal of A^G. Then the Brauer quotient is $A(G) = A^G / \sum_{H<G} A_H^G$.
So, the Brauer homomorphism on $A = I(P, F)$ with respect to H is

$$Br_H : A^H \longrightarrow A(H), \qquad f \longrightarrow f + A_{<H}^H \qquad \text{where} f \in A^H.$$

Theorem 2.3.8. Let G be a finite group. Let P be a locally finite poset which is a G-poset. Let $I(P, F)$ be the modular incidence G-algebra over F. Then $I(P, F)^G = Z\big(I(P, F)\big)$.

Definition 2.3.9. Let G be a finite group. Let P be a locally finite poset which is a G-poset. Let $I(P, F)$ be the modular incidence algebra of P over F. The modular incidence algebra $I(P, F)$ is G-algebra over F, by the action $f^a(x, y) = f(x^a, y^a)$ for all $f \in I(P, F)$ and $a \in G$. We have $\delta \in I(P, F)$ as well $\delta^a \in I(P, F)$. We have a group homomorphism $\varphi : G \longrightarrow U\big(I(P, F)\big)$ defined by $\varphi(a) = \delta^a$ for all $a \in G$. The modular incidence algebra $I(P, F)$ is called interior G-algebra over F.

Chapter 3

Tensor product of incidence algebras

In this chapter, we present the notion of tensor product. In Section 3.1, we study tensor product of vector spaces. We have compiled some basic theorems and we summarize without proofs. In Section 3.2, we shall study the tensor product of two algebras A and B over F. We then prove $A \otimes_F B$ is an algebra over F. If G_1 and G_2 are finite groups and A is a G_1-algebra, B is a G_2-algebra we can prove $A \otimes_F B$ is a $G_1 \times G_2$-algebra. We shall prove $A \otimes_F B$ is an interior $G_1 \times G_2$-algebra. In Section 3.3, we work on an uncountable locally partial order set. We define the cartesian product of two posets. We then mention the cartesian product of two uncountable posets is also uncountable. If $I(P_1, F)$ and $I(P_2, F)$ are two incidence algebras we shall prove the tensor product $I(P_1, F) \otimes_F I(P_2, F)$ is an incidence algebra. We then prove that the incidence algebra $I(P_1 \times P_2, F)$ is isomorphic to the incidence algebra $I(P_1, F) \otimes_F I(P_2, F)$.

3.1 Tensor product of vector spaces

In this section, we will define the direct product of two vector spaces over the same field. We will define a linear map and a bilinear map. We will define the tensor product of two vector spaces, can be seen more in [3, 5, 7] and [10].

Definition 3.1.1. Let F be a field. Let V and U be vector spaces over F. The direct product $V \times U$ is defined by

$$V \times U = \{(v, u) : v \in V \quad \text{and} \quad u \in U\}.$$

The addition on $V \times U$ can be defined as

$$(v_1, u_1) + (v_2, u_2) = (v_1 + v_2, u_1 + u_2),$$

for all $(v_1, u_1), (v_2, u_2) \in V \times U$.
The scalar multiplication on $V \times U$ can be defined as

$$\alpha(v, u) = (\alpha v, \alpha u),$$

for all $(v, u) \in V \times U$ and $\alpha \in F$.

Lemma 3.1.2. Let V and U be two vector spaces over F. Then the direct product $V \times U$ is a vector space over F.

Lemma 3.1.3. Let V and U be two vector spaces over F which are finite dimensional. Then the direct product $V \times U$ is finite dimensional and $dim(V \times U) = dimV + dimU$.

Definition 3.1.4. Let V and U be two vector spaces over the field F. A linear map is a function $f : V \longrightarrow U$ such that for all $v_1, v_2 \in V$ and $\alpha \in F$ the following two conditions are satisfied:

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$

$$f(\alpha v_1) = \alpha f(v_1).$$

Definition 3.1.5. If F is a field. Given three vector spaces V, U and W over the field F. A function $f : V \times U \longrightarrow W$ is said to be a bilinear map if satisfies these properties:

$$f(v_1 + v_2, u) = f(v_1, u) + f(v_2, u)$$

$$f(v, u_1 + u_2) = f(v, u_1) + f(v, u_2)$$

$$f(\alpha v, u) = \alpha f(v, u) = f(v, \alpha u).$$

On other hand, a function $f : V \times U \longrightarrow W$ is a bilinear map if it is linear in each variable, this mean f has the following properties:

- The map $f_u : V \longrightarrow W$ is a linear map such that $v \longmapsto f(v, u)$, for any fixed element $u \in U$
- The map $f_v : U \longrightarrow W$ is a linear map such that $u \longmapsto f(v, u)$, for any fixed element $v \in V$.

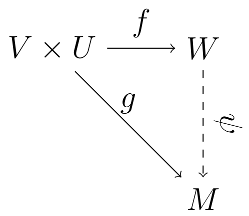
For more details of the construction of tensor product, the reader can see the following references ([5] page 71, [7] page 207, [10] page 46)

Theorem 3.1.6. Let V and U be two vector spaces over F.

(1) There exist a vector space W over F and a bilinear map $f : V \times U \longrightarrow W$ which satisfy the following conditions

(a) W is generated by the image $f(V \times U)$ of f.

(b) If M is a vector space over F and $g : V \times U \longrightarrow M$ is a bilinear map, there exists an unique linear map $\psi : W \longrightarrow M$ such that $g = \psi \circ f$.



(2) If W' is a vector space over F and $f' : V \times U \longrightarrow W'$ is a bilinear map satisfy conditions (a) and (b), then there exists an unique isomorphism $j : W \longrightarrow W'$ such that $j \circ f = f'$.

Definition 3.1.7. Let V and U be vector spaces over F. A tensor product of V and U is a pair (W, f) consisting of a vector space W over F and a bilinear map $f : V \times U \longrightarrow W$ which is satisfy the conditions (a) and (b) in Theorem 3.1.6. We write $W = V \otimes U$ and $f(v, u) = v \otimes u$. The map f is called the canonical of a tensor product $W = V \otimes U$. The tensor product $W = V \otimes U$ is generated by $\{v \otimes u : \quad v \in V, \quad u \in U\}$.

Furthermore, for every $v, v_1, v_2 \in V$ and $u, u_1, u_2 \in U$ and $\alpha \in F$ we have

$$(v_1 + v_2) \otimes u = v_1 \otimes u + v_2 \otimes u$$

$$v \otimes (u_1 + u_2) = v \otimes u_1 + v \otimes u_2$$

$$(\alpha v) \otimes u = \alpha(v \otimes u) = v \otimes (\alpha u).$$

The uniqueness property (2) of Theorem 3.1.6 can be restated as follows

Theorem 3.1.8. If (W_1, f_1) and (W_2, f_2) are tensor products of V and U, then there exists an unique linear isomorphism $\psi : W_1 \longrightarrow W_2$ with $\psi \circ f_1 = f_2$.

3.2 Tensor product of algebras

In this section, we shall define the tensor product of two algebras over a field. We explain that the tensor product of algebras is an algebra. If F is a field. Suppose that G_1 and G_2 are two finite groups, if A is a G_1-algebra and B is a G_2-algebra, we prove that the tensor product $A \otimes_F B$ is carries a $G_1 \times G_2$-algebra structure. We will explain when the tensor product of two algebras will be considered as interior $G_1 \times G_2$-algebra, can be seen more in [14].

Definition 3.2.1. Let A and B be two algebras over F. The tensor product $A \otimes_F B$ is given by

$$A \otimes_F B = \{\sum_i (a_i \otimes_F b_i) : a_i \in A \quad \text{and} \quad b_i \in B\}.$$

The product of two elements in $A \otimes_F B$ as the form

$$\big(\sum_i (a_i \otimes_F b_i)\big)\big(\sum_j (a'_j \otimes_F b'_j) = \sum_{i,j} a_i a'_j \otimes_F b_i b'_j,$$

where $a_i, a'_j \in A$ and $b_i, b'_j \in B$.

In the following theorems, we prove it on the basis of tensor product and thus it will be correct on all elements.

Theorem 3.2.2. *The tensor product of two algebras over F is an algebra over F.*

Proof. Let A and B be algebras over F. Firstly, since A and B are vector spaces over F hence the tensor product of A and B is a vector space over F. Secondly, we prove that the tensor product $A \otimes_F B$ is a ring

(1) $(A \otimes_F B, +)$ is an abelian group.

(2) $(A \otimes_F B, \cdot)$ is a monoid such that

- Associative property of multiplication

$$\begin{aligned}\big((a_1 \otimes_F b_1)(a_2 \otimes_F b_2)\big)(a_3 \otimes_F b_3) &= (a_1 a_2 \otimes_F b_1 b_2)(a_3 \otimes_F b_3)\\ &= \big((a_1 a_2) a_3\big) \otimes_F \big((b_1 b_2) b_3\big)\\ &= \big(a_1 (a_2 a_3)\big) \otimes_F \big(b_1 (b_2 b_3)\big)\\ &= (a_1 \otimes_F b_1)(a_2 a_3 \otimes_F b_2 b_3)\\ &= (a_1 \otimes_F b_1)\big((a_2 \otimes_F b_2)(a_3 \otimes_F b_3)\big),\end{aligned}$$

for all $a_1 \otimes_F b_1, a_2 \otimes_F b_2, a_3 \otimes_F b_3 \in A \otimes_F B$.

- There is an identity element $1_A \otimes_F 1_B$ in $A \otimes_F B$ such that

$$(a \otimes_F b)(1_A \otimes_F 1_B) = a1_A \otimes_F b1_B = a \otimes_F b$$
$$(1_A \otimes_F 1_B)(a \otimes_F b) = 1_A a \otimes_F 1_B b = a \otimes_F b,$$

for all $a \otimes_F b \in A \otimes_F B$.

(3) Distributive property of multiplication over addition

$$\begin{aligned}(a_1 \otimes_F b_1)\big((a_2 \otimes_F b_2) + (a_3 \otimes_F b_3)\big) &= (a_1 \otimes_F b_1)(a_2 + a_3 \otimes_F b_2 + b_3)\\ &= a_1(a_2 + a_3) \otimes_F b_1(b_2 + b_3)\\ &= (a_1a_2 + a_1a_3) \otimes_F (b_1b_2 + b_1b_3)\\ &= (a_1a_2 \otimes_F b_1b_2) + (a_1a_3 \otimes_F b_1b_3)\\ &= (a_1 \otimes_F b_1)(a_2 \otimes_F b_2) + (a_1 \otimes_F b_1)(a_3 \otimes_F b_3)\end{aligned}$$

$$\begin{aligned}\big((a_1 \otimes_F b_1) + (a_2 \otimes_F b_2)\big)(a_3 \otimes_F b_3) &= (a_1 + a_2 \otimes_F b_1 + b_2)(a_3 \otimes_F b_3)\\ &= (a_1 + a_2)a_3 \otimes_F (b_1 + b_2)b_3\\ &= (a_1a_3 + a_2a_3) \otimes_F (b_1b_3 + b_2b_3)\\ &= (a_1a_3 \otimes_F b_1b_3) + (a_2a_3 \otimes_F b_2b_3)\\ &= (a_1 \otimes_F b_1)(a_3 \otimes_F b_3) + (a_2 \otimes_F b_2)(a_3 \otimes_F b_3).\end{aligned}$$

Thirdly, for all $a_1 \otimes_F b_1, a_2 \otimes_F b_2 \in A \otimes_F B$ and $\alpha \in F$

$$\begin{aligned}\alpha\big((a_1 \otimes_F b_1)(a_2 \otimes_F b_2)\big) &= \alpha(a_1a_2 \otimes_F b_1b_2)\\ &= \big(\alpha(a_1a_2)\big) \otimes_F b_1b_2\\ &= \big((\alpha a_1)a_2\big) \otimes_F b_1b_2\\ &= \big((\alpha a_1) \otimes_F b_1\big)(a_2 \otimes_F b_2)\\ &= \big(\alpha(a_1 \otimes_F b_1)\big)(a_2 \otimes_F b_2)\end{aligned}$$

$$\begin{aligned}\alpha\big((a_1 \otimes_F b_1)(a_2 \otimes_F b_2)\big) &= \alpha(a_1a_2 \otimes_F b_1b_2)\\ &= a_1a_2 \otimes_F \big(\alpha(b_1b_2)\big)\\ &= a_1a_2 \otimes_F \big(b_1(\alpha b_2)\big)\\ &= (a_1 \otimes_F b_1)\big(a_2 \otimes_F (\alpha b_2)\big)\\ &= (a_1 \otimes_F b_1)\big(\alpha(a_2 \otimes_F b_2)\big).\end{aligned}$$

We get

$$\alpha\big((a_1 \otimes_F b_1)(a_2 \otimes_F b_2)\big) = \big(\alpha(a_1 \otimes_F b_1)\big)(a_2 \otimes_F b_2) = (a_1 \otimes_F b_1)\big(\alpha(a_2 \otimes_F b_2)\big).$$

Therefore the tensor product $A \otimes_F B$ is an algebra over F.

□

Theorem 3.2.3. If G_1 and G_2 are finite groups. Given A and B are G_1-algebra over F and G_2-algebra over F respectively. Then the tensor product $A \otimes_F B$ is a $G_1 \times G_2$-algebra over F.

Proof. Suppose that A is a G_1-algebra over F and B is a G_2-algebra over F. The group $G_1 \times G_2$ acts on the tensor product $A \otimes_F B$ by $(a \otimes b)^{(g_1,g_2)} = a^{g_1} \otimes_F b^{g_2}$ for all $(g_1, g_2) \in G_1 \times G_2$ and $a \otimes_F b \in A \otimes_F B$. To check that

$$(a \otimes_F b)^{(e_{G_1},e_{G_2})} = a^{e_{G_1}} \otimes_F b^{e_{G_2}} = a \otimes_F b$$

$$\begin{aligned}\big((a \otimes_F b)^{(h_1,h_2)}\big)^{(g_1,g_2)} &= (a^{h_1} \otimes_F b^{h_2})^{(g_1,g_2)} \\ &= (a^{h_1})^{g_1} \otimes_F (b^{h_2})^{g_2} \\ &= a^{h_1 g_1} \otimes_F b^{h_2 g_2} \\ &= \big(a \otimes_F b\big)^{(h_1 g_1, h_2 g_2)}\end{aligned}$$

$$\begin{aligned}\big((a_1 \otimes_F b_1) + (a_2 \otimes_F b_2)\big)^{(g_1,g_2)} &= \big((a_1 + a_2) \otimes_F (b_1 + b_2)\big)^{(g_1,g_2)} \\ &= (a_1 + a_2)^{g_1} \otimes_F (b_1 + b_2)^{g_2} \\ &= (a_1^{g_1} + a_2^{g_1}) \otimes_F (b_1^{g_2} + b_2^{g_2}) \\ &= (a_1^{g_1} \otimes_F b_1^{g_2}) + (a_2^{g_1} \otimes_F b_2^{g_2}) \\ &= (a_1 \otimes_F b_1)^{(g_1,g_2)} + (a_2 \otimes_F b_2)^{(g_1,g_2)}\end{aligned}$$

$$\begin{aligned}\big((a_1 \otimes_F b_1)(a_2 \otimes_F b_2)\big)^{(g_1,g_2)} &= \big((a_1 a_2) \otimes_F (b_1 b_2)\big)^{(g_1,g_2)} \\ &= (a_1 a_2)^{g_1} \otimes_F (b_1 b_2)^{g_2} \\ &= (a_1^{g_1} a_2^{g_1}) \otimes_F (b_1^{g_2} b_2^{g_2}) \\ &= (a_1^{g_1} \otimes_F b_1^{g_2})(a_2^{g_1} \otimes_F b_2^{g_2}) \\ &= (a_1 \otimes_F b_1)^{(g_1,g_2)}(a_2 \otimes_F b_2)^{(g_1,g_2)}\end{aligned}$$

$$\begin{aligned}\big(\alpha(a \otimes_F b)\big)^{(g_1,g_2)} &= \big((\alpha a) \otimes_F b\big)^{(g_1,g_2)} \\ &= (\alpha a)^{g_1} \otimes_F b^{g_2} \\ &= (\alpha (a)^{g_1}) \otimes_F b^{g_2} \\ &= \alpha(a^{g_1} \otimes_F b^{g_2}) \\ &= \alpha(a \otimes_F b)^{(g_1,g_2)}.\end{aligned}$$

□

Remarks.

- The tensor product $A \otimes_F B$ which is a $G_1 \times G_2$-algebra is called external tensor product.
- If A and B are G-algebras then the tensor product $A \otimes_F B$ is called internal tensor product.

Theorem 3.2.4. Let G_1 and G_2 be two finite groups. Let A be an interior G_1-algebra and B be an interior G_2-algebra then the tensor product $A \otimes_F B$ is also an interior $G_1 \times G_2$-algebra.

Proof. Suppose A is an interior G_1-algebra and B is an interior G_2-algebra then there exist two group homomorphisms
$\phi_1 : G_1 \longrightarrow U(A)$ is given by $\phi_1(g_1) = g_1 \cdot 1_A$, for all $g_1 \in G_1$
$\phi_2 : G_2 \longrightarrow U(B)$ is given by $\phi_1(g_2) = g_2 \cdot 1_B$, for all $g_2 \in G_2$.
The tensor product $A \otimes_F B$ is an interior $G_1 \times G_2$-algebra if there is a group homomorphism. We assume that a function $\phi : G_1 \times G_2 \longrightarrow U(A \otimes_F B)$ is given by $\phi\big((g_1, g_2)\big) = (g_1 \cdot 1_A) \otimes_F (g_2 \cdot 1_B)$ for all $(g_1, g_2) \in G_1 \times G_2$. Now, we prove that ϕ is a group homomorphism as follows

$$\begin{aligned}
\phi\big((g_1, g_2)(h_1, h_2)\big) &= \phi\big((g_1 h_1, g_2 h_2)\big) \\
&= (g_1 h_1) \cdot 1_A \otimes_F (g_2 h_2) \cdot 1_B \\
&= (g_1 h_1) \cdot 1_A 1_A \otimes_F (g_2 h_2) \cdot 1_B 1_B \\
&= (g_1 \cdot 1_A)(h_1 \cdot 1_A) \otimes (g_2 \cdot 1_B)(h_2 \cdot 1_B) \\
&= \big((g_1 \cdot 1_A) \otimes_F (g_2 \cdot 1_B)\big)\big((h_1 \cdot 1_A) \otimes_F (h_2 \cdot 1_B)\big) \\
&= \phi\big((g_1, g_2)\big)\phi\big((h_1, h_2)\big).
\end{aligned}$$

Furthermore, the tensor product $A \otimes_F B$ is an interior $G_1 \times G_2$-algebra. $\square$

3.3 Tensor product of incidence algebras

The tensor product of incidence algebras appeared in the paper Ahmad Alghamdi [2] and deal with finite partially ordered sets. We will try to deal with uncountable locally partially ordered sets. We prove that the tensor product of two incidence algebras is an incidence algebra.

Definition 3.3.1. A poset P is uncountable if it is an infinite poset which contains too many elements to be countable.

We deal with another definition of uncountable poset.

Definition 3.3.2. A poset P is uncountable if there is no injective function from P to the natural numbers.

Definition 3.3.3. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two posets. The cartesian product of P_1 and P_2 is again a poset

$$P_1 \times P_2 = \{(x, y) : x \in P_1 \quad \text{and} \quad y \in P_2\}.$$

With relation $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq_1 x_2$ and $y_1 \leq_2 y_2$ where $x_1, x_2 \in P_1$ and $y_1, y_2 \in P_2$.

Remark. Certainly, the cartesian product of two finite locally posets is a locally finite poset because if every interval of P_1 and P_2 are finite thus every interval of $P_1 \times P_2$ is finite.

Lemma 3.3.4. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be uncountable locally posets. Then the cartesian product $(P_1 \times P_2, \leq)$ is uncountable.

Since the cartesian product $P_1 \times P_2$ is a locally poset then there is an incidence algebra of $P_1 \times P_2$ over F as the following proposition.

Proposition 3.3.5. Let $(P_1, \leq_1)$ and $(P_1, \leq_2)$ be uncountable locally posets. Let $(P_1 \times P_2, \leq)$ be a cartesian product of P_1 and P_2 which is an uncountable locally poset. Then there is an incidence algebra $I(P_1 \times P_2, F)$ over F, its elements has the form $f_{P_1 \times P_2}\big((x_1, x_2), (y_1, y_2)\big)$.

Proof. If $f_{P_1} : P_1 \times P_1 \longrightarrow F$ is an incidence function in $I(P_1, F)$ and $f_{P_2} : P_2 \times P_2 \longrightarrow F$ is an incidence function in $I(P_1, F)$ we have

$$f_{P_1}(x_1, y_1) = 0_F \quad \text{whenever} \quad x_1 \nleq_1 y_1, \quad x_1, y_1 \in P_1$$

$$f_{P_2}(x_2, y_2) = 0_F \quad \text{whenever} \quad x_2 \nleq_2 y_2, \quad x_2, y_2 \in P_2.$$

We take $\big((x_1,x_2),(y_1,y_2)\big) \in (P_1 \times P_2)^2$. Hence
$f_{P_1\times P_2}\big((x_1,x_2),(y_1,y_2)\big) = f_{P_1}(x_1,y_1)\cdot f_{P_2}(x_2,y_2) = 0_F$
whenever $(x_1,x_2) \nleq (y_1,y_2)$. We get the incidence algebra as following

$$I(P_1\times P_2,F) = \{f_{P_1\times P_2} : (P_1\times P_2)^2 \longrightarrow F : f_{P_1\times P_2}\big((x_1,x_2),(y_1,y_2)\big) = 0_F \quad \text{if}$$

$$(x_1,x_2) \nleq (y_1,y_2)\}.$$

It follows that the incidence function $f_{P_1\times P_2}\big((x_1,x_2),(y_1,y_2)\big)$ is an element of the incidence algebra $I(P_1 \times P_2, F)$ over F. □

In the following proposition we prove that the tensor product of two incidence algebras is an incidence algebra.

Proposition 3.3.6. Let $(P_1,\leq_1)$ and $(P_2,\leq_2)$ be uncountable locally posets. Let $I(P_1,F)$ and $I(P_2,F)$ be incidence algebras over F. Then the tensor product $I(P_1,F)\otimes_F I(P_2,F)$ is an incidence algebra over F and its elements has the form $\sum_i (f_{P_i} \otimes_F f'_{P_i})$ where $f_{P_i} \in I(P_1,F), f'_{P_i} \in I(P_2,F)$.

Proof. Suppose that $I(P_1,F)$ is an incidence algebra over F, the incidence function $f_{P_1} \in I(P_1,F)$ where $f_{P_1}(x_1,y_1) = 0_F$ if $x_1 \nleq_1 y_1$, $x_1,y_1 \in P_1$ and suppose that $I(P_2,F)$ is an incidence algebra over F, the incidence function $f_{P_2} \in I(P_2,F)$ where $f_{P_2}(x_2,y_2) = 0_F$ if $x_2 \nleq_2 y_2$, $x_2,y_2 \in P_2$. We need show that $I(P_1,F)\otimes_F I(P_2,F)$ is an incidence algebra, we have

$$\big(f_{P_1}\otimes_F f_{P_2}\big)\big((x_1,x_2),(y_1,y_2)\big) = f_{P_1}(x_1,y_1)\otimes_F f_{P_2}(x_2,y_2) = 0_F \otimes_F 0_F = 0_F$$

if $(x_1,x_2) \nleq (y_1,y_2)$. Therefore the tensor product of incidence algebras $I(P_1,F)\otimes_F I(P_2,F)$ is an incidence algebra as following

$$I(P_1,F)\otimes_F I(P_2,F) = \{f_{P_1}\otimes_F f_{P_2} : (P_1\times P_2)^2 \longrightarrow F :$$

$$\big(f_{P_1}\otimes_F f_{P_2}\big)\big((x_1,x_2),(y_1,y_2)\big) = 0_F \quad \text{if} \quad (x_1,x_2) \nleq (y_1,y_2)\}.$$

□

Theorem 3.3.7. Let F be algebraically closed field. Let $(P_1,\leq_1)$ and $(P_2,\leq_2$) be uncountable locally posets. Let $I(P_1,F)$ and $I(P_2,F)$ be incidence algebras over F. Let $(P_1\times P_2,\leq)$ be a cartesian product of P_1 and P_2 which is an uncountable locally poset. Then $I(P_1\times P_2,F) \cong I(P_1,F)\otimes_F I(P_2,F)$.

Proof. Define a function $\phi : I(P_1 \times P_2, F) \longrightarrow I(P_1, F) \otimes_F I(P_2, F)$ by

$$\phi(f_{P_1 \times P_2}) = f_{P_1} \otimes_F f_{P_2}$$

$$\phi(f_{P_1 \times P_2}(x, y)) = \big(f_{P_1} \otimes_F f_{P_2}\big)(x, y),$$

where $x = (x_1, x_2) \in P_1 \times P_2$ and $y = (y_1, y_2) \in P_1 \times P_2$.
For all $f_{P_1 \times P_2}, f'_{P_1 \times P_2} \in I(P_1 \times P_2)$ and $\alpha \in F$. We prove that ϕ is a homomorphism

$$\begin{aligned}
\phi(\alpha f_{P_1 \times P_2}) &= (\alpha f_{P_1} \otimes_F f_{P_2}) \\
&= \alpha(f_{P_1} \otimes_F f_{P_2}) \\
&= \alpha\phi(f_{P_1 \times P_2}) \\
\phi\big(f_{P_1 \times P_2} + f'_{P_1 \times P_2}\big) &= \big(f_{P_1} + f'_{P_1}\big) \otimes_F \big(f_{P_2} + f'_{P_2}\big) \\
&= \big(f_{P_1} \otimes_F f_{P_2}\big) + \big(f'_{P_1} \otimes_F f'_{P_2}\big) \\
&= \phi\big(f_{P_1 \times P_2}\big) + \phi\big(f'_{P_1 \times P_2}\big) \\
\phi\big(f_{P_1 \times P_2} * f'_{P_1 \times P_2}\big) &= \big(f_{P_1} * f'_{P_1}\big) \otimes_F \big(f_{P_2} * f'_{P_2}\big) \\
&= \big(f_{P_1} \otimes_F f_{P_2}\big) * \big(f'_{P_1} \otimes_F f'_{P_2}\big) \\
&= \phi\big(f_{P_1 \times P_2}\big) * \phi\big(f'_{P_1 \times P_2}\big) \\
\phi(\delta_{P_1 \times P_2}) &= \delta_{P_1} \otimes_F \delta_{P_2}.
\end{aligned}$$

The function ϕ is injective because:

$$\begin{aligned}
\phi\big(f_{P_1 \times P_2}(x, y)\big) &= \phi\big(f'_{P_1 \times P_2}(x, y)\big) \\
\big(f_{P_1} \otimes_F f_{P_2}\big)(x, y) &= \big(f'_{P_1} \otimes_F f'_{P_2}\big)(x, y) \\
f_{P_1}(x_1, y_1) \otimes_F f_{P_2}(x_2, y_2) &= f'_{P_1}(x_1, y_1) \otimes_F f'_{P_2}(x_2, y_2) \\
f_{P_1}(x_1, y_1) \cdot f_{P_2}(x_2, y_2) &= f'_{P_1}(x_1, y_1) \cdot f'_{P_2}(x_2, y_2) \\
f_{P_1 \times P_2}(x, y) &= f'_{P_1 \times P_2}(x, y).
\end{aligned}$$

The function ϕ is surjective because: for all $f \in I(P_1, F) \otimes_F I(P_2, F)$, there is $f_{P_1} \in I(P_1, F)$ and $f_{P_2} \in I(P_2, F)$ such that $f = f_{P_1} \otimes_F f_{P_2}$. Hence, there is $f_{P_1 \times P_2} \in I(P_1 \times P_2, F)$ such that $\phi(f_{P_1 \times P_2}) = f_{P_1} \otimes_F f_{P_2} = f$. Therefore $I(P_1 \times P_2, F) \cong I(P_1, F) \otimes_F I(P_2, F)$. □

Chapter 4

Blocks of incidence algebras

In this chapter, we defined into three sections. In Section 4.1, we study the decomposition of a modular incidence algebra $I(P, F)$ into the algebras $e_iI(P, F)$ which are called block incidence algebra. As well as the block algebra $e_iI(P, F)$ is decomposed into indecomposable $I(P, F)$-modules, where e_i is a central primitive idempotent. Previously in Section 1.3 we defined a trace map and studied it in order to define a defect group. In Section 4.2, we will present main concepts in the theory of G-algebras, these concepts are a pointed group, a projective relative, a local pointed group, a defect pointed group and a nilpotent block of a modular incidence algebra. In Section 4.3, we briefly present category theory and use it to make important connection among group algebra and incidence algebra.

4.1 Block algebra and defect group of incidence algebras

In this section, we will learn how to decomposed the modular incidence algebra. We will define a block algebra of a modular incidence algebra and define a defect group of a block of incidence algebra. We shall follow [9, 10].

Let P be a locally finite poset. Let $I(P, F)$ be the modular incidence algebra of P over F which is finite dimensional. Decompose the modular incidence algebra $I(P, F)$ as a direct sum

$$I(P, F) = B_1 \oplus B_2 \oplus ... \oplus B_t$$

of the decomposable as the F-algebras $B_i, 1 \leq i \leq t$. In the following definition we will study the structure of an algebra $B_i, 1 \leq i \leq t$.

Definition 4.1.1. Let P be a locally finite poset. Let $I(P, F)$ be the modular incidence algebra of P over F. A block idempotent of $I(P, F)$ is a primitive idempotent e_i in $Z(I(P,F))$. The algebra $B_i = I(P,F)e_i$ is called a block incidence algebra of $I(P,F)$.

$$1_{I(P,F)} = 1_{Z(I(P,F))} = e_1 + e_2 + ... + e_t.$$

Proposition 4.1.2. Let P be a locally finite poset. Let $I(P,F)$ be the modular incidence algebra. Then $I(P,F)$ can be decomposed into direct sum of finite number of blocks, each of which is a two sided ideal of $I(P,F)$.

Proof. Suppose that $I(P,F)$ is the modular incidence algebra of a locally finite poset P over F. We have, the modular incidence algebra $I(P,F)$ is a finite dimensional. So the identity element for the modular incidence algebra can be decomposed into direct sum of finite number of central primitive idempotent, as follows

$$1_{Z(I,F)} = 1_{I(P,F)} = e_1 + e_2 + ... + e_t.$$

Hence, we have

$$I(P,F) \cdot 1_{I(P,F)} = I(P,F)(e_1 + e_2 + ... + e_t)$$

$$I(P,F) = I(P,F)e_1 \oplus I(P,F)e_2 \oplus ... \oplus I(P,F)e_t.$$

Since e_i is central $\forall i$,

$$I(P,F) = e_1 I(P,F) \oplus e_2 I(P,F) \oplus ... \oplus e_t I(P,F).$$

Therefore, $I(P,F)$ is decomposed into direct sum of finite number of two sided ideals. Where the $I(P,F)e_i$ are two-sided ideals of $I(P,F)$. These are called blocks of $I(P,F)$. □

Corollary 4.1.3. Let P be a locally finite poset. Let $I(P,F)$ be the modular incidence algebra of P over F. Then the incidence algebra $I(P,F)$ is decomposed into Hecke-algebras $e_i \cdot I(P,F) \cdot e_i$, where e_i is a central primitive idempotent.

Proof. Since e_i is central, we have

$$e_i I(P,F) = I(P,F)e_i$$

$$e_i e_i I(P,F) = e_i I(P,F)e_i.$$

Since e_i is idempotent, hence

$$e_i I(P,F) = e_i I(P,F)e_i, \qquad \forall i \in \{1,...,t\}.$$

We know that the incidence algebra $I(P,F)$ is decomposed into block incidence algebras $e_i I(P,F)$. There are two orthogonal e_i and e_j in $I(P,F)$ for all i,j distinct, such that $e_i \cdot e_j = 0$ and

1. $e_i \cdot I(P,F) \cap e_j \cdot I(P,F) = \{0\}$
2. $\sum_i e_i \cdot I(P,F) = I(P,F)$.

We get that

$$I(P,F) = e_1 I(P,F) e_1 \oplus e_2 I(P,F) e_2 \oplus ... \oplus e_t I(P,F) e_t.$$

As required. □

Corollary 4.1.4. Let P be a locally finite poset. Let $I(P,F)$ be the modular incidence algebra. Let e be a central primitive idempotent in $I(P,F)$ then

$$I(P,F) = (1-e)I(P,F) \oplus eI(P,F).$$

Theorem 4.1.5. Let A be a finite dimensional algebra over F and f be an idempotent in A. Then f is primitive if and only if fA is an indecomposable A-module.

Theorem 4.1.6. Let P be a locally finite poset. Let $I(P,F)$ be the modular incidence algebra. If e_i is a central primitive idempotent such that $e_i = f_1 + f_2 + ... + f_r$ where f_j is a primitive idempotent which is not central. Then the block incidence algebra $e_i I(P,F)$ is decomposed into indecomposable $I(P,F)$-module $f_j I(P,F)$.

Proof. Suppose that e_i is a central primitive idempotent and $e_i = f_1 + f_2 + ... + f_r$ is an idempotent decomposition, where f_j is a primitive idempotent which is not central. The block incidence algebra $e_i I(P,F)$ can be decomposed as follows

$$\begin{aligned} e_i I(P,F) &= (f_1 + f_2 + ... + f_r) I(P,F) \\ &= f_1 I(P,F) \oplus f_2 I(P,F) \oplus ... \oplus f_r I(P,F). \end{aligned}$$

Since f_j is primitive, $1 \leq j \leq r$ then $f_j I(P,F)$ is an indecomposable $I(P,F)$-module, By Theorem 4.1.5. Hence the block incidence algebra $e_i I(P,F)$ is decomposed into indecomposable $I(P,F)$-module. □

Theorem 4.1.7. Let P be a locally finite poset. Let $I(P,F)$ be the modular incidence algebra. Let $e \in Z(I(P,F))$ and $eI(P,F)$ be a block incidence algebra. Then $Z(eI(P,F))$ is a local algebra over F.

In Chapter 1, Section 1.3 we studied the trace map t_H^G and we mentioned its properties. Now, we use it in definition a defect group in the following.

Definition 4.1.8. Let G be a finite group. Let P be a locally finite poset which is a G-poset. Let $I(P,F)$ be the modular incidence G-algebra of P over F. Let e be a block idempotent of $I(P,F)$ and let $I(P,F)e$ be a block incidence algebra of $I(P,F)$. A defect group of the block $I(P,F)e$ is a p-subgroup D of G with the properties: the idempotent e belongs to the ideal $I(P,F)_D^G$ and if there exists a subgroup H such that the idempotent e belongs to the ideal $I(P,F)_H^G$ then D is a G-conjugate to a subgroup of H.

Remarks.

- Defect groups of p-blocks are p-groups.
- The defect groups of the principal p-block are the Sylow p-subgroups.
- p-block of defect zero has the trivial subgroup as a defect group.
- The intersections of two Sylow p-subgroups are defect groups.
- Each normal p-subgroup is contained in a defect group.
- Defect theory is a generalization of Sylow theory.

4.2 Pointed group and nilpotent blocks

In this section, we shall define a pointed group on a modular incidence algebra. We define a projective relative and a local pointed group. We define a defect pointed group. We will define a nilpotent block. We present some examples. We shall follow [9, 14].

Definition 4.2.1. Let A be an algebra over F. Let g and f be idempotents in A. The elements g and f are called associate in A if there are elements x and y in A such that $xy = g$ and $yx = f$.

Remark:

- It is easy to prove that $g \cdot A \cdot f$ and $f \cdot A \cdot g$ are F-subalgebras of A.
- The algebra $gAf = 0$ if g and f are orthogonal and central.
- We have the elements $a = gxf \in gAf$ and $b = fyg \in fAg$ satisfy:

$$ab = gxffyg = gxfyg = gxyxyg = g^4 = g$$
$$ba = fyggxf = fygxf = fyxyxf = f^4 = f.$$

Lemma 4.2.2. The relation associate in A is an equivalence relation. If g and f are associate in A we say that $g \backsim f \Leftrightarrow \exists \quad x, y \in A$ such that $xy = g$ and $yx = f$.

Proof.

1. If $f \backsim f$. There are $x, y \in A$ such that $xy = f$ and $yx = f$. Then there are elements $a = fxf \in fAf$ and $b = fyf \in fAf$ such that $ab = f$ and $ba = f$. Hence it is reflexive.
2. If $g \backsim f$, g and f are associate in A thus there are $x, y \in A$ such that $xy = g$ and $yx = f$. Then there are elements $a = gxf \in gAf$ and $b = fyg \in fAg$ such that $ab = g$ and $ba = f$
   It is clear that $f \backsim g$. Hence it is symmetric.
3. If $g \backsim f$ and $f \backsim h$, f and h are associate in A thus there are $z, w \in A$ such that $zw = f$ and $wz = h$. Then there are elements $c = fzh \in fAh$ and $d = hwf \in hAf$ such that $cd = f$ and $dc = h$. Hence We conclude that

$$acdb = afb = gxfffyg = gxfyg = ab = g$$
$$dbac = dfc = hwfffzh = hwfzh = hwzwzh = h^4 = h.$$

We have $ac = gxzh \in gAh$ and $db = hwyg \in hAg$. Then there are $xz, wy \in A$ such that $xzwy = g$ and $wyxz = h$. Hence g and h are associate in A and $g \backsim h$. Hence the relation is transitive.

□

Definition 4.2.3. Let A be an algebra over F. The points of A is the classes of associate primitive idempotents in A and we denote the set of points of A by $P(A)$.

Definition 4.2.4. Let G be a finite group. Let P be a locally finite poset which is a G-poset. Let $I(P, F)$ be the modular incidence G-algebra of P over F. A pointed group on $I(P, F)$ is a pair $(H, \beta) = H_\beta$ where H is a subgroup of G and β is a point of $I(P, F)^H$.

Example 4.2.5. If $G = S_3$, we take $P = \{\{1\}, A_3, S_3\}$ to be a locally finite poset which is a G-poset. We take $F = \mathbb{Z}_2$. The incidence algebra of P over $\mathbb{Z}_2$ is

$$I(P, \mathbb{Z}_2) = \{f : P \times P \longrightarrow \mathbb{Z}_2, f(K, L) = 0 \quad \text{if} \quad K \nleq L\}.$$

If $H = \{(1), (12)\}$ is a subgroup of G. We note that for all $K \in P$ then $K^h = K$ for all $h \in H$. The set of H-fixed points of $I(P, \mathbb{Z}_2)$ is

$$\big(I(P, \mathbb{Z}_2)\big)^H = \{f \in I(P, \mathbb{Z}_2), \quad f^h(K, L) = f(K, L), \text{for all} \quad h \in H \quad \& \quad K, L \in P\}.$$

The pair $(H, \mu) = H_\mu$ is a pointed group on $I(P, \mathbb{Z}_2)$ where the möbius function μ is a point of $\big(I(P, \mathbb{Z}_2)\big)^H$ since for all $h \in H$

$$\begin{aligned} \mu^h(\{1\}, S_3) &= \mu(\{1\}^h, S_3^h) = \mu(\{1\}, S_3) \\ \mu^h(\{1\}, A_3) &= \mu(\{1\}^h, A_3^h) = \mu(\{1\}, A_3) \\ \mu^h(A_3, S_3) &= \mu(A_3^h, S_3^h) = \mu(A_3, S_3). \end{aligned}$$

So $\mu^h(K, L) = \mu(K, L)$ for all $h \in H$ and $K, L \in P$.
For μ to be idempotent, we see that:

$$(\mu * \mu)(x, y) = \sum_{x \leq z \leq y} \mu(x, z)\mu(z, y) = \mu(x, y).$$

We have the maximal ideal M_β of $I(P, F)^H$, with simple quotient being $A(H_\beta) = I(P, F)^H / M_\beta$. Then the quotient map

$$Br_\beta : I(P, F)^H \longrightarrow A(H_\beta), \qquad f \longrightarrow f + M_\beta$$

is another Brauer homomorphism.

Remark. There are two Brauer homomorphisms:

$$Br_H : I(P,F)^H \longrightarrow A(H), \qquad f \longrightarrow f + A^H_{<H}$$

$$Br_\beta : I(P,F)^H \longrightarrow A(H_\beta), \qquad f \longrightarrow f + M_\beta.$$

The Brauer homomorphisms Br_H is identical to the Brauer homomorphisms Br_β if there is a homomorphism ϕ is define by

$$\phi : A(H) \longrightarrow A(H_\beta). \quad \phi(f + A^H_{<H}) = f + M_\beta,$$

such that $Br_\beta = \phi \circ Br_H$.

Definition 4.2.6. If H_β and K_α are two pointed groups on a modular incidence algebra $I(P,F)$ and satisfying $K \subseteq H$. We say that K_α is a pointed subgroup of H_β, ($K_\alpha \leq H_\beta$) if:

- There are elements $i \in \beta$ and $j \in \alpha$ such that $jI(P,F)j \subseteq iI(P,F)i$.
- For any element $g \in \beta$ there exists an element $f \in \alpha$ such that $fI(P,F)f \subseteq gI(P,F)g$.
- $Br_\alpha(\beta) \neq 0$.

Definition 4.2.7. Let G be a finite group. Let P be a locally finite poset which is a G-poset. Let $I(P,F)$ be the modular incidence algebra which is G-algebra. Let H and K be two subgroups of G and $K \leq H$. We have the relative trace map $t^H_K : I(P,F)^K \longrightarrow I(P,F)^H$. Given two pointed groups H_β and K_α on $I(P,F)$ we say that H_β is projective relative to K_α if

1. $K \leq H$.
2. $\beta \subseteq t^H_K(I(P,F)^K \alpha I(P,F)^K)$.

We write $H_\beta pr K_\alpha$.

Remark. We can say that H_β is projective relative to K_α if satisfying $I(P,F)^H \beta I(P,F)^H \subseteq t^H_K(I(P,F)^K \alpha I(P,F)^K)$.

Definition 4.2.8. A pointed group Q_δ on a modular incidence G-algebra $I(P,F)$ is a local pointed group if it has any one of the following equivalent properties:

- Q_δ is minimal with respect to the relation pr.
- Q_δ is not projective relative to a proper subgroup of Q.

- $\delta \nsubseteq I(P,F)_R^Q$ for every proper subgroup R of Q.
- $Br_Q(\delta) \neq 0$.
- $Br_\delta(I(P,F)_R^Q) = 0$ for every subgroup R of Q.
- $Ker(Br_Q) \subseteq M_\delta$.

Definition 4.2.9. Let G be a finite group. Let P a locally finite poset. Let $I(P,F)$ be the modular incidence G-algebra. Let H_β and K_α be two pointed groups on $I(P,F)$. We say that K_α is a defect pointed subgroup of H_β if

1. $K_\alpha \leq H_\beta$.
2. $H_\beta pr K_\alpha$.
3. K_α is a local.

Lemma 4.2.10. The local pointed subgroup K_α of H_β is a defect pointed subgroup of H_β if $I(P,F)^H \beta I(P,F)^H \subseteq t_K^H(I(P,F)^K \alpha I(P,F)^K)$.

Definition 4.2.11. Let G be a finite group. Let P a locally finite poset which is a G-poset. Let $I(P,F)$ be the modular incidence G-algebra. A block b of $I(P,F)$ is said to be nilpotent if the quotient group $N_G(Q_\delta)/C_G(Q)$ is a p-group for any local pointed subgroup Q_δ of G_α, where α is a point of G on $I(P,F)$ containing b. We say that the block algebra $I(P,F)b$ is a nilpotent block algebra of $I(P,F)$.

We shall illustrate the definition by some examples.

Example 4.2.12. If $p = 2$. We take the field $\mathbb{Z}_2$ of characteristic 2. Let $G = D_8 = \langle a, x : a^4 = e, x^2 = e, (ax)^2 = e \rangle$ be the dihedral group of order 8. Consider the locally finite G-poset $P = \{\{e\}, \langle a^2 \rangle, \langle a^2, x \rangle, D_8\}$. Let $I(P,\mathbb{Z}_2)$ be the modular incidence algebra of P over $\mathbb{Z}_2$ which is the G-algebra. We studied the modular incidence algebra of P over $\mathbb{Z}_2$ in section 2.2, Example 2.2.1 and we will link it to nilpotent block algebra. The pair (G,μ) is a pointed group on $I(P,\mathbb{Z}_2)$ where μ is a point of $\big(I(P,\mathbb{Z}_2)\big)^G$. We take $Q = \langle a^2 \rangle = \{e, a^2\}$ to be a subgroup of D_8. The pair $(Q,\delta) = Q_\delta$ is a pointed group on $I(P,\mathbb{Z}_2)$ where δ is a point of $\big(I(P,\mathbb{Z}_2)\big)^Q$. The normalizer of Q_δ in the group D_8 is equal to D_8 and the centralizer of Q of G is equal to D_8. So the block μ is the nilpotent since the quotient group $N_G(Q_\delta)/C_G(Q)$ is the trivial group which is a p-group. The block algebra $I(P,\mathbb{Z}_2)\mu$ is the nilpotent block algebra of $I(P,\mathbb{Z}_2)$.

Example 4.2.13. Given the field $\mathbb{Z}_3$ of characteristic 3 where $p = 3$. Let $E = \langle x, y, z | \quad x^3 = y^3 = z^3 = e, [x, y] = z, \quad [x, z] = [z, y] = e \rangle$ be the extra special group of order 27. Consider the locally finite E-poset $P = \{\{e\}, \langle z \rangle, \langle z, y \rangle, E\}$. Let $I(P, \mathbb{Z}_3)$ be the modular incidence algebra of P over $\mathbb{Z}_3$ which is the E-algebra. We studied the modular incidence algebra of P over $\mathbb{Z}_3$ in section 2.2, Example 2.2.2 and we will link it to nilpotent block algebra. The pair (E, μ) is a pointed group on $I(P, \mathbb{Z}_3)$ where μ is a point of $\big(I(P, \mathbb{Z}_3)\big)^E$. We take $Q = \langle z \rangle = \{e, z, z^2\}$ to be a subgroup of E. The pair $(Q, \delta) = Q_\delta$ is a pointed group on $I(P, \mathbb{Z}_3)$ where δ is a point of $\big(I(P, \mathbb{Z}_3)\big)^Q$. We note that $N_E(Q_\delta) = E$ and $C_E(Q) = E$ hence the quotient group $N_E(Q_\delta)/C_E(Q)$ is the trivial group which is a p-group. We get the block μ is the nilpotent and the block algebra $I(P, \mathbb{Z}_3)\mu$ is the nilpotent block algebra of $I(P, \mathbb{Z}_3)$.

Example 4.2.14. If $p = 2$. we take the field $\mathbb{Z}_2$. Let $G = S_3$ be the symmetric group. Consider the locally finite G-poset $P = \{\{1\}, A_3, S_3\}$. Let $I(P, \mathbb{Z}_2)$ be the incidence algebra of P over $\mathbb{Z}_2$ which is the G-algebra. The pair (G, ζ) is a pointed group on $I(P, \mathbb{Z}_2)$ where ζ is a point of $\big(I(P, \mathbb{Z}_2)\big)^G$. We take $Q = \big\{\{1\}\big\}$ to be the trivial subgroup of S_3. The pair $(Q, \lambda) = Q_\lambda$ is a pointed group on $I(P, \mathbb{Z}_2)$ where λ is a point of $\big(I(P, \mathbb{Z}_2)\big)^Q$. We note that $N_G(Q_\lambda) = G$ and $C_G(Q) = G$. So the block ζ is the nilpotent since the quotient group $N_G(Q_\lambda)/C_G(Q)$ is the trivial group which is a p-group and the block algebra $I(P, \mathbb{Z}_2)\zeta$ is the nilpotent block algebra of $I(P, \mathbb{Z}_2)$.

4.3 Category algebra and the relation between group algebra and incidence algebra

In this section, we define a category and give some examples, can be seen in [7]. We will offer two definitions for the category algebra over a field which has characteristic prime number. These two definitions agree. We then explain the relationship between group algebra and incidence algebra.

Definition 4.3.1. A category $\mathcal{C}$ consists of a class of objects and a class of morphisms such that following conditions must be satisfied.

(a) For all A, B objects of $\mathcal{C}$ there is a set of morphisms $Mor(A, B)$ $f : A \longrightarrow B$ such that

$$Mor(A, B) \cap Mor(A', B') = \varnothing \qquad \text{if} \quad (A, B) \neq (A', B').$$

(b) For each A, B, C objects of $\mathcal{C}$ there is a rule of composition

$$Mor(A, B) \times Mor(B, C) \longrightarrow Mor(A, C)$$

such that if $(g, f) \mapsto gf$ then

- Associativity: if $f : A \longrightarrow B, g : B \longrightarrow C$ and $h : C \longrightarrow D$ are morphisms in $\mathcal{C}$ then $(hg)f = h(gf)$.
- Identity: for each A object of $\mathcal{C}$ there is an identity morphism $I_A : A \longrightarrow A$ such that $fI_A = f$ and $I_A g = g$ for any morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow A$ of a category $\mathcal{C}$.

Remarks.

- A morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is said to be an equivalence if there is a morphism $g : B \longrightarrow A$ in $\mathcal{C}$ such that $gf = I_A$ and $fg = I_B$.
- The composite of two equivalences is an equivalence.
- If $f : A \longrightarrow B$ is an equivalence, we say that A and B are equivalent.

Example 4.3.2. A partially ordered set P can be considered as a category. The objects of a category P are their elements. If $x, y \in P$ we can write $f : x \longrightarrow y$ to indicate that $x \leq y$. The set of morphism $Mor(x, y)$ consists of one morphism $f : x \longrightarrow y$ if $x \leq y$ and $Mor(x, y) = \varnothing$ if $x \nleq y$. The rule of composition is given by the transitive property of the partial order: if $f : x \longrightarrow y$ and $g : y \longrightarrow z$ then $gf : x \longrightarrow z$ since $x \leq y$ and $y \leq z$ imply $x \leq z$.

Example 4.3.3. Let $\mathcal{C}$ be a class of all groups. If $G, H \in \mathcal{C}$ then the morphism $Mor(G, H)$ is the set of all group homomorphism $f : G \longrightarrow H$. The rule composition is the composition of group homomorphisms.

Definition 4.3.4. Let $\mathcal{C}$ be a category with a morphism set. Let F be a field. The category algebra $F\mathcal{C}$ of $\mathcal{C}$ over F is a vector space with basis $Mor(\mathcal{C})$. In other words, $F\mathcal{C}$ consists of formal linear combinations of the form $\sum a_i\varphi_i$, where $\varphi_i \in Mor(\mathcal{C})$ and $a_i \in F$. Define a multiplication operation on $F\mathcal{C}$ as follows

$$\sum a_i\varphi_i \sum b_j\psi_j = \sum a_ib_j(\varphi_i \circ \psi_j).$$

Definition 4.3.5. The category algebra $F\mathcal{C}$ of $\mathcal{C}$ over F consists of all functions $f : Mor(\mathcal{C}) \longrightarrow F$. The multiplication is described by a convolution, if $f, g \in F\mathcal{C}$ then the product $f * g$ is defined as

$$(f * g)(\alpha) = \sum_{\varphi\circ\psi=\alpha} f(\varphi)g(\psi).$$

Example 4.3.6. Let F be a field. Let $\mathcal{C}$ be a group, the group is the same as a category with a single object. The morphism are endomorphism, where the elements of the group correspond to the morphisms of the category. By Definition 4.3.4, the category algebra $F\mathcal{C}$ as the group algebra.

Example 4.3.7. Let F be a field. Let $\mathcal{C} = P$ be a partially ordered set. For any pair of objects x, y there is at most one morphism from x to y. By Definition 4.3.5, the category algebra FP of P over F is $f : Mor(P) \longrightarrow F$. Then the category algebra FP as the incidence algebra.

Remark. The groups and the posets are special kinds of categories. Similarly, the group algebra and the incidence algebra are special cases of a category algebra. In fact, the incidence algebra is similar to the group algebra. Where we mean by similar that the algebraic structures as well as the behaviour of the invariants in each category of them look like the same. This enables us to see the properties in both sides and hence to get parallel results in each case.

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