Chapter 1

Introduction

The famous conjecture of Kaplansky (see [3, Pages 55-76]) says that:

Given two unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ semi-simple. Does any surjective linear application $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves invertibility (i.e $\phi(x)$ is invertible in $\mathcal{B}$ for any invertible element $x \in \mathcal{A}$) a Jordan morphism?

Note that this formulation of the problem is due to Aupetit [5]. This conjuncture has been solved in many cases:

(i) If $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional, [28].

(ii) If $\mathcal{B}$ is a commutative Banach algebra, [5].

(iii) If $\mathcal{A} = \mathcal{B}(X)$, $\mathcal{B} = \mathcal{B}(Y)$ where X, Y are two Banach spaces, [31, 26].

(iv) If $\mathcal{A}$ and $\mathcal{B}$ are two von Neumann algebras, [6].

(v) If $\mathcal{B}$ has a separating family of finite dimensional irreducible representations, [15].

But the problem remains unsolved even for C*-algebras. The interest reader may consult [5, 11, 26] for more details.

On the other hand, inspired by this conjecture, an important fields of research is the problem of describing maps on operators and matrices that preserve certain functions, subsets and relations has been widely studied in the literature, see [7], [9], [10], [11], [14], [15], [16], [17], [23], [29], [30], [32], [33] and their references therein. One of the classical problems in this area of research is to characterize maps preserving the spectra of the product of operators. Molnár in [29] studied maps preserving the spectrum of operator and matrix products. His results have been extended in several directions [8], [1], [2], [12], [13], [19], [21], [22], [24] and [25]. In [2], the problem of characterizing maps between matrix algebras preserving the spectrum of polynomial products of matrices is considered. In particular, the results obtained therein extend and unify several results obtained in [11] and [13].

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex infinite dimensional Hilbert spaces. Let $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}(\mathcal{K})$) denote the algebra of all bounded linear operators on $\mathcal{H}$ (resp. on $\mathcal{K}$). We say that a map $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ preserves the skew Lie product of operators if

$$[\varphi(T), \varphi(S)]_* = [T, S]_*$$

where $[T, S]_* = TS - ST^*$ for any operators $S, T \in \mathcal{B}(\mathcal{H})$. Latter in [1], the form of all maps preserving the spectrum and the local spectrum of skew Lie product of matrices are determined.

In this thesis we will examine the form of surjective maps preserving the spectrum of skew Lie product of operators on an infinite dimensional complex Hilbert space. The plan is as follows

(i) Some of the basic definitions and terminology used in this thesis are introduced in Chapter 2.

(ii) In Chapter 3, maps preserving peripheral spectrum of Jordan products of self-adjoint operators are discussed

(iii) In the last chapter, we deal with the problem of characterizing surjective maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ preserves the skew Lie product of operators. Precisely, we shall prove the following.

Theorem 1.0.1. *A surjective map* $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *satisfies*

$$\sigma\left(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*\right) = \sigma\left(TS - ST^*\right), \ (T, \ S \in \mathcal{B}(\mathcal{H})), \tag{1.1}$$

if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ *such that*

$$\varphi(T) = \pm UTU^* \tag{1.2}$$

for all $T \in \mathcal{B}(\mathcal{H})$.