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ON THE PSEUDOBLOCKS OF FINITE DIMENSIONAL ALGEBRAS

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Abstract

We introduce the concept of pseudoblock for finite dimensional algebras, and investigate the pseudoblock structure for various finite dimensional algebras such as semisimple algebras, the group algebras of cyclic groups, and the triangular algebras. Towards the end, we determine the pseudoblocks for the group algebra $FSL(2,p)$ in characteristic p, which turns out to be identical with the Brauer blocks.

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Contents

Notations

A	The finite dimension F-algebra.
$F = \mathbb{F}_p$	The field in characteristic prime number p.
$\mathbb{N}$	The set of nonnegative integers (natural numbers).
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The rational field.
Λ	The group algebra FG.
${}_AA$	The left regular module (as left A-module).
$M_n(F)$	The algebra of all $n \times n$ matrices over a field F.
$\underset{PSA}{\approx}$	The pseudoblock linkage principle on $IndA$.
$\underset{A}{\approx}$	The Brauer linkage principle on $modA$.
$B \otimes C$	The tensor producte of B and C over the field F.
$\oplus$	The internal direct sum of moduls.
$P \vert A$	P is dircte summand of A.
$a \vert b$	a divides b.
$a \nmid b$	a does not divide b.
$\vert G : H \vert$	The index of a subgroup H in a group G.
$\vert G \vert$	The order of the group G.
D^t	The transpose of the matrix D.
$f \circ g$	The composition of maps.
$\cong$	Is isomorphic to.
$\sim$ and $\equiv$	Is equivalent to
$gcd(p, m)$	The greatest common divisor of p and m.
I_n	The identity matrix $n \times n$.
1_A	The identity element of A.
F_G	The trivial module.
St_G	The Steinberg representation.
$CharF$	The characteristic of F.
$\dim\Lambda$	The dimension of Λ.
$IndA$	The set of indecomposable A-module.
$IndsY$	The set of isomorphism classes of indecomposable direct A-summands of Y.
$modA$	The category of finite dimensional left A-modules.
$Hom_A(X, Y) = (X, Y)_A$	All morphism from A-module X to A-module Y.
$Mor_{MmodA}(X, Y)$	The space of morphisms from X to Y.
$E = End_A(Y)$	Endomorphism algebra.
$Irr(E(Y))$	The set of isomorphism classes of irreducible $E(Y)$-module.
$modF$	The category of vector space over F.
$MmodA$	The category whose objects are the covariant functors.
$ind_H^G L = L^G$	The induction FG-module.
$res_H^G(M) = M\vert_H = M_H$	The restriction FH-module.
PIM	The projective indecomposable A-module.
$vx(M)$	The vertices of the indecomposable FG-module M.

$GL(m,p)$	The general linear group.
$SL(m,p)$	The special linear group.
$PSL(m,p)$	The projective special linear group.
FS_3	The symmetric group algebra.
FC_n	The group algebra of cyclic groups.
$FSL(2,p)$	The group algebra of special linear group.
$rad(A)$	The radical of A.
$soc(A)$	The socal of A.
ACC	The ascending chain condition.
DCC	The descending chain condition.
$Kerg$	The kernel of g.
Img	The image of g.
d.v.r	The discrete valuation ring.
$Syl_p(G)$	The Sylow p-subgroup of G.

INTRODUCTION

The pseudoblock is the branch of block theory that it partitions the set of classes of indecomposable modules in a very useful way. The origins of pseudoblock were in the paper of Ahmed A. Khammash " The Pseudoblocks of Endomorphism Algebras", written in 2009, [16]. In this work, he introduced the notion of pseudoblock of the endomorphism algebra $E(Y) = End_A(Y)$ and showed the compatibility of the pseudoblock distribution of the indecomposable A-summands of Y with the block distribution of the simple $E_A(Y)$-modules. In 2014, Ahmed A. Khammash [17] studied compatibility between the pseudoblock of endomorphism algebras and the tensor product, and he related the Brauer-Fitting correspondence as well as the notion of pseudoblocks of endomorphism algebras to the tensor product of modules and algebras.

In this dissertation, we borrow the notion "pseudoblock" from [16] to introduce it to finite dimensional (not necessary endomorphism) algebras. We shall investigate the pseudoblock structure of several finite dimensional algebras. The dissertation is organized as follows:

Chapter 0 is a background chapter collects all basic notions and results which are needed for this dissertation.
Chapter 1, we introduce the concept of pseudoblocks of finite dimensional algebras A, and we explain the concept of the Brauer linkage principle of finite dimensional algebras, then we borrow the notion of pseudoblocks of an endomorphism algebra of a module and introduce it for any finite dimensional algebras in the light of the Brauer-Fitting correspondence, where we find that the pseudoblocks for the indecomposable A-summand of Y is compatible with the block for the simple $End_A(Y)$-module. Also, we introduce criteria simplifies the determination of the pseudoblock linkage principle.
Chapter 2, we discuss the connection between the Brauer linkage principle $\underset{A}{\approx}$ and the pseudoblock linkage principle $\underset{PSA}{\approx}$, where we find that the pseudoblock linkage principle is stronger than the Brauer linkage principle.
Chapter 3, we revise the concept of tensor product of algebras and modules. Then we prove that the notion of pseudoblocks is compatible with the tensor product. We also study compatibility between the tensor product and the indecomposable A-module, and Brauer linkage principle.
Chapter 4, we discuss the pseudo-block distribution of the indecomposable modules for some various finite dimensional algebras such as semisimple algebras, the triangular algebra A, the symmetric group algebra FS_3 in all characteristics, cyclic group algebra over a field of characteristic prime number p, and p-group algebra.
Chapter 5, we determine the pseudoblock structure of the group algebra of special linear group $\Lambda = FG; G = SL(2,p)$ in characteristic prime number p. We have chosen the group algebra $\Lambda = FSL(2,p)$, because this is the only finite group of Lie type, which is of finite representation type; i.e. Λ has finite indecomposable modules. Furthermore, we introduce the complete set of projective indecomposable Λ-module as stated in [2], and then we describe the complete set of indecomposable Λ-module as stated in the paper of D. Craven [4], where we use the Green Correspondence theory, which relates the isomorphism classes of indecomposable FG-modules with isomorphism classes of indecomposable $FN_G(U)$-modules; So that we can use theorem (0.3.11). Also, we find the block theory of Λ-modules as stated in [9].

Finally, we determine the pseudoblocks of the group algebra $\Lambda = FSL(2,p)$ in characteristics p, also we study the pseudoblocks of group algebra $FSL(2,p)$ in characteristics $p = 2, 3$ and 5, and then compare the block and pseudoblock theory of the group algebra Λ in characteristics p.

Chapter 0

PRELIMINARIES

We devote this chapter to recall some necessary background for the next chapters. Here, we introduce the most important basic concepts in finite dimensional algebras.

0.1 ALGEBRAS AND MODULES

<u>ALGEBRAS AND GROUP ALGEBRAS</u>

First, we introduce the concept of algebra, and we recall its basic properties.

DEFINITION 0.1.1. *([13], p.227). Let F be a field. A is an algebra over the field F or (A is an F-algebra) iff*

1. *A is a ring;*

2. *A is a vector space over F;*

3. *let $\lambda \in F, \forall a, b \in A$. Then $(\lambda a)b = \lambda(ab) = a(\lambda b)$.*

Furthermore, if A is a finite dimension, then A is called a finite dimension algebra over the field F.

Note: The basic concepts of linear algebra can be taken from the following book [13].

EXAMPLE 0.1.2. *([13], p.227). Every a field $(F, +, .)$ is an algebra over itself, in which $(F, +, .)$ is a ring, $(F, +, .)$ is a vector space, and $\lambda \in F, \forall \alpha, \beta \in F$ then $\lambda(\alpha\beta) = (\lambda\alpha)\beta = \alpha(\lambda\beta)$.*

See, examples in ([13], p.227).

DEFINITION 0.1.3. *([10], p.13). Let F be a field, and let A, B be F-algebras. A map $f : A \longrightarrow B$ is an F-algebra homomorphism (or f is an algebra map). if*

1. $f(a + b) = f(a) + f(b)$;

2. $f(ab) = f(a)f(b)$;

3. $f(\lambda a) = \lambda f(a)$;

4. $f(1_A) = 1_B$, $\forall a, b \in A, \lambda \in F$.

Moreover, F-algebra homomorphism $Hom_F(X, X) = End_F(X)$ is endomorphism algebra.

Also in this dissertation, we need to talk about group algebra, where group algebra is an F-algebra.

DEFINITION 0.1.4. *([19], p.42). Let F be a field, and let G be a finite group, where $G = \{g_1, g_2, ..., g_d : |G| = d\}$. Then $FG = \{\sum_{i=1}^{d} \alpha_i g_i : \alpha_i \in F, g_i \in G\}$ is the group algebra of G over F, which satisfying the following;*

1. $\sum_{i=1}^{d} \alpha_i g_i + \sum_{i=1}^{d} \alpha_i' g_i = \sum_{i=1}^{d} (\alpha_i + \alpha_i') g_i$;
2. $(\sum_{i=1}^{d} \alpha_i g_i)(\sum_{j=1}^{d} \alpha_j g_j) = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i \alpha_j g_i g_j = \sum_{k=1}^{d} (\sum_{i,j}^{d} \alpha_i \alpha_j) g_k$;
3. $\lambda(\sum_{i=1}^{d} \alpha_i g_i) = \sum_{i=1}^{d} (\lambda \alpha_i) g_i$, *where* $\lambda \in F$.

Furthermore, if group algebra FG is finite dimension, then the dimension of FG is equal to the order of the group G.

EXAMPLE 0.1.5. *([10], p.6). Let F be a field, and let G be the cyclic group of order 3, in which $G =< x >$, i.e. $G = \{x, x^2, 1\}$. Then $FG = \{r_0 + r_1 x + r_2 x^2 : \forall r_0, r_1, r_2 \in F$ & $x, x^2, 1 \in G\}$ is group algebra, where FG satisfies the definition (0.1.4).*

More details on algebra are given in [10, 13].

MODULES AND REPRESENTATIONS

A vector space over a field F is an A-module, where A is an F-algebra. Also, algebra A and A-module are equivalent. So, we introduce the concept of A-module and some of its properties

DEFINITION 0.1.6. *([19], p.8). Let G be a group. The vector space V over the field F is called FG-module if a multiplication $vx, (v \in V, x, y \in G)$ is defined, such that:*

(i) $vx \in V$;

(ii) $(hv + kw)x = h(vx) + k(wx), \quad (v, w \in V, \quad h, k \in F)$;

(iii) $v(xy) = (vx)y$;

(iv) $1v = v$.

See, examples in ([10], section 2.1).

DEFINITION 0.1.7. *([19], p.10). Let V be an A-module. Then U is a submodule of V if*

- *U is vector subspace of V, (i.e. $U \subset V$);*
- *U is an A-module, i.e. $ux \in U$, $\forall u \in U$ and $\forall x \in G$;*

in which if $\{0_V\} \subsetneqq U \subsetneqq V$ is an A-submodule of V. Then V is called reducible over F; otherwise it is called (simple) irreducible over F.

DEFINITION 0.1.8. *([19], p.23). Let V be an A-module. Then V is semisimple (completely reducible) if $V = U_1 \oplus U_2 \oplus \ldots \oplus U_l$, where U_i; $\forall i = 1, 2, \ldots, l$ are (irreducible) simple A-modules.*

DEFINITION 0.1.9. *([13], p.181). Let R be a ring with identity. Then the unitary module H is a free R-module if it satisfies at least one of the following condition:*

1. *H has a non-empty basis;*

2. *H is the internal direct sum of a family of cyclic R-module, each of which is isomorphic as a left R-module to R;*

3. *H is R-module isomorphic to a direct sum of copies of the left R-module R.*

In ([19], p.8), we find that an FG-module is equivalent to the matrix representation. Hence,

DEFINITION 0.1.10. *([19], p.3) A matrix representation over a field F of degree $m \in \mathbb{N}$ for a group G is a group homomorphism $\rho : G \to GL(m, F)$, where $GL(m, F)$ is the general linear group of degree m over F.*

See, examples in ([19], p.18).

REMARK.

1. *General linear group $GL(m, F)$ is the set of all non-singular $m \times m$ matrices with coefficients in a given field F, in([19], p.3).*

2. *$G = SL(m, p)$ is special linear group, where $SL(m, p) = \{X \in GL(m, F) : detX = 1, CharF = p\}$ from ([7], p.75).*

3. *It is well-known (see [19]) that the notions of matrix representation of a group G over a field F (or algebra A) and FG-modules (A-modules) are equivalent. All modules considered in this dissertation will be left modules.*

DEFINITION 0.1.11. *([19],p.4). If $g : G \longrightarrow GL(m, F)$ is injective, then g is called faithful representation.*

Now, we recall some concepts of induced modules.

DEFINITION 0.1.12. *([6], p.228). Let FG be a group algebra, let M be a left FG-module, and let H be a subgroup of G (i.e. $FH \subseteq FG$). Then the operation of the restriction of scalars from group algebra FG to FH assigns to each left FG-module M a left FH-module $res^G_H(M)$; we denote it by $M|_H$ or M_H.*

DEFINITION 0.1.13. *([6], p.228) Let FG be a group algebra, let H be a subgroup of group G where $FH \subseteq FG$, and let L be a left FH-module. Then the operation of induction from FH-modules to FG-modules assigns to each left FH-module L a left FG-module $ind^G_H(L)$ given by*

$$L^G = ind^G_H(L) = FG \otimes_{FH} L.$$

i.e. if L is matrix representation of H over a field F, then L^G denotes the matrix representation of G afforded by L.

The following theorem shows the dimension of induction L^G.

THEOREM 0.1.14. *([2], p.56). Let L be a left FH-module, and let L^G be induction from FH-modules to FG-modules. Then*

$$dim\ ind^G_H(L) = |G : H|\ dim(L).$$

More details on induced modules are given in [6].

THEOREM 0.1.15. *(Lifting Process) ([19], p.58) . Let N be a normal subgroup of G and let $A_0(Nx)$ is a representation of degree m of the group G/N. Then*

$$A(x) = A_0(Nx)$$

defines a representation of G, lifted from G/N.
Moreover, if M is an FG-module, M can be regarded as $F(G/N)$-module, where $mx = m(Nx); \forall x \in G, m \in M$ is well-defined of G/N on M, hence N acts trivially on M.

Now, we introduce some notions related to an FG-module homomorphism.

DEFINITION 0.1.16. *([19], p.24). Let F be a field, let G be a group, and let FG be a group algebra, and let X, Y be an FG-modules. Then $\theta : X \to Y$ is an FG-module homomorphism $Hom_{FG}(X, Y) = (X, Y)_{FG}$ if:*

1. $\theta(x + y) = \theta(x) + \theta(y), \quad \forall x, y \in X;$
2. $\theta(\lambda x) = \lambda\theta(x), \quad \forall x \in X, \lambda \in F;$
3. $\theta(xg) = (\theta x)g, \quad \forall x \in X, g \in G.$

DEFINITION 0.1.17. *([19]). Let F be a field, let G be a group, and let FG be a group algebra, let X be an FG-module, and let $\theta : X \longrightarrow X$ be an FG-module homomorphism. Then θ is an FG-module endomorphism.*

$$End_{FG}(X) = \{\theta : X \longrightarrow X \mid \theta \text{ is } FG\text{-map}\}.$$

DEFINITION 0.1.18. *([13], p.31). Let A be an F-algebra, and let $g : X \longrightarrow Y$ be an A-module homomorphism.*

1. *The kernel of g is $kerg = \{a \in X \mid g(a) = 0_Y \in Y\}$, where 0_Y is the identity in Y.*
2. *The image of g is $Img = \{b \in Y \mid \exists a \in X, g(a) = b\}$.*

From important theorems of algebra that has been used the First Isomorphism Theorem:

THEOREM 0.1.19. *(First Isomorphism Theorem) ([13], p.44). Let A be an F-algebra, and let $g : X \longrightarrow B$ be an A-module homomorphism. Then $\dfrac{X}{kerg} \cong Img$.*

More details on modules and representation theory are given in [10, 19].
Now, we introduce some concepts of p-modular system.

DEFINITION 0.1.20. *([6], p.402). Let K be a field of characteristic zero, let R be a discrete valuation ring (d.v.r), and let F be a field of characteristic prime number p, where $F = R/radR$; $radR$ is radical R. Then the system (K, R, F) is called p-modular system.*

More details on p-modular system and discrete valuation ring are given in [6, Chapter2] and [6, p.81].

From concepts that we need it a lot in this dissertation are the decomposition matrix and the Cartan matrix.

DEFINITION 0.1.21. *([3], p.17). Let F be a field, let G be a group, and let FG be a group algebra. The decomposition matrix $D = (d_{ij})_{r\times s}$ has rows indexed by the ordinary characters χ_i; $1 \leq i \leq r$ of G and column indexed by the modular characters ϕ_i; $1 \leq i \leq s$ of G, and the entry d_{ij} is the multiplicity of ϕ_j in the modular reduction of χ_i. It is known that $DD^t = C = (c_{ij})_{s\times s}$; the Cartan matrix, where c_{ij} is the multiplicity of ϕ_i as a composition factor of the projective cover of ϕ_j.*

REMARK. *Let $\Lambda = FG$ be a group algebra. Then*

1. *Let St_G be denote the Steinberg module for group G, see ([14], p.300).*
2. *The Steinberg module is the unique largest irreducible Λ-module, and also St_G is projective indecomposable Λ-module as stated in ([12], p.256, p.259).*
3. *The dimension of St_G is equal to the order of a Sylow p-subgroup as stated in ([12], p.250).*
4. *The diagonal form of decomposition matrix D corresponding to the blocks of FG, see ([11], section16.3).*

One of the most important concepts that we need it a lot in this dissertation is p-regular. Hence,

DEFINITION 0.1.22. *([5], p.283). Let p be a prime number. An element $g \in G$ is p-regular if p does not divide the order of g. An element whose order is a power of p is called p-singular.*
Consequently, a conjugate class C in G is p-regular if all its elements are p-regular.

The following theorem shows, how do we know the number of simple modules?

THEOREM 0.1.23. *([2], p.14). Let F be a field, let G be a group, and let FG be a group algebra. Then the number of simple FG-modules equals the number of p-regular conjugacy classes of G.*

EXAMPLE 0.1.24. *Let F be a field of characteristic p, let $G = S_3 =< a, b|a^3 = b^2 = 1, bab^{-1} = a^{-1} >$, and let $C_1 = \{1\}$, $C_2 = \{a, a^2\}$, and $C_3 = \{b, ab, a^2b\}$ be conjugacy classes. If $p = 2$, then the group algebra FG has 2 2-regular conjugacy classes of G; it is C_1 and C_2; and if $p = 3$, then FG has 2 3-regular conjugacy classes of G; it is C_1 and C_3.*

0.2 INDECOMPOSABLE MODULES

Indecomposable modules important type of modules. Here, we introduce the concept of indecomposable modules, and some of its theorems. Details can be found in [2, 5].

DEFINITION 0.2.1. *([5], p.81). Let A be an F-algebra. An A-module U is indecomposable if $U \neq 0$ and if it cannot be written as a direct sum of two non-trivial submodules (i.e. $U \neq b_1 \oplus b_2$, where b_1, b_2 are non-trivial submodules). Otherwise, it is said to be decomposable.*

EXAMPLE 0.2.2. *$\mathbb{Z}$-module $(\mathbb{Z}_3, +_3)$ is indecomposable $\mathbb{Z}$-module, while $\mathbb{Z}_6 = \mathbb{Z}_3 \oplus \mathbb{Z}_2$ is decomposable $\mathbb{Z}$-module.*

We introduce the concept of idempotent, nilpotent, orthogonal, and primitive elements.

DEFINITION 0.2.3. *([3], p.11). Let A be an F-algebra, and let x, y be a non-zero element in A. x is idempotent if $x^2 = x$, also $1 - x$ is idempotent.*

If there exists a positive integer n such that $y^n = 0$, then y is nilpotent.

Two idempotents x_1, x_2 are orthogonal if $x_1x_2 = x_2x_1 = 0$.

An idempotent x is primitive if we cannot write $x = x_1 + x_2$, with x_1, x_2 are orthogonal idempotents.

Now, we introduce the concept of local algebra as follows:

DEFINITION 0.2.4. *([2], p.21). An F-algebra A is local algebra (or local) if and only if every element of A is nilpotent or invertible.*

The following lemma shows that, local algebra has only 0 and the identity idempotents.

LEMMA 0.2.5. *([10], p.135). Let E be a local F-algebra. Then the only idempotents in E are 0 and the identity 1_E.*

Proof. Let E be a local F-algebra. By definition (0.2.4), for all elements in E is nilpotent or invertible. Let $e \in E$ be an idempotent, (i.e. $e^2 = e$). Then $1_E - e$ is also idempotent. If e has inverse, and also $(1_E - e)$ has inverse. Let $x \in E$, where $x(1_E - e) = 1_E$.
Then $e = 1_E e = x(1_E - e)e = xe - xe^2 = 0$, this gives $e = 0$.
Also, There exists $a \in E$ such that $ae = 1_E$. Then $e = 1_E e = aee = ae^2 = ae = 1_E$, this gives $e = 1_E$.
Accordingly, the only idempotents in E are 0 and 1_E. □

The endomorphism algebra of U is local, if and only if U is indecomposable A-module, as follows:

THEOREM 0.2.6. *([2], p.22). Let A be an F-algebra. The A-module U is indecomposable if and only if $End(U)$ is local.*

DEFINITION 0.2.7. *([5], p.340). Let F be a field, let G be a group, and let FG be a group algebra, let M be a left FG-module, and let $M_1, M_2, \ldots, M_r$, where r in $\mathbb{N}$ be indecomposable FG-modules component of M, i.e.*

$$M = m_1M_1 \oplus m_2M_2 \oplus \ldots \oplus m_rM_r.$$

Then M is multiplicity free FG-module if all $M_1, M_2, \ldots, M_r$ appear exactly once, i.e. $m_1 = m_2 = \ldots = m_r = 1$.

The identity element of group algebra FG can be written uniquely as a sum of commuting primitive idempotents as follows:

THEOREM 0.2.8. *([11], p.67). Let F be a field, let G be a group, let $\Lambda = FG$ be a group algebra, let 1_Λ be the identity element of Λ, and let $1_\Lambda = e_1 \oplus e_2 \oplus \ldots \oplus e_n$, where e_i; $1 \leq i \leq n$ be a primitive idempotents. Then $1_\Lambda = e_1 \oplus e_2 \oplus \ldots \oplus e_n$ is unique.*

Now, we recall the definition of projective indecomposable A-module. From good references for projective indecomposable modules are [3, 13]

DEFINITION 0.2.9. *([3], p.6). Let A be an F-algebra, let P be an A-module, and let W, V be any two A-modules. There are $\lambda : P \longrightarrow V$ and $\mu : W \longrightarrow V$, where μ is epimorphism (i.e. μ is an A-module homomorphism & surjective), and there is $\nu : P \longrightarrow W$. Then P is projective. It is shown in Figure (1). Moreover, if $\lambda : P \longrightarrow V$ is an essential epimorphism, then P is projective cover for V.*

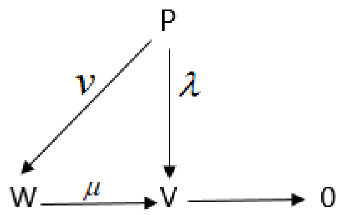

Figure 1:

REMARK.

1. *P is a projective if and only if P is a direct summand of a free module ([13], p.192).*
2. *Let A be an F-algebra, and let N be an A-module, also we have an A-module epimorphism $\mu : W \to V$. Then A-module homomorphism $\lambda : V \to N$ is essential if λ is surjective, and if for each sequence of A-modules $W \xrightarrow{\mu} V \xrightarrow{\lambda} N$ such that $\lambda\mu$ is surjective ([6], p.131).*

EXAMPLE 0.2.10. *([13], p.193). Let $\mathbb{Z}_3$, $\mathbb{Z}_2$ be $\mathbb{Z}_6$-modules. There is $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules, because $\mathbb{Z}_3$ & $\mathbb{Z}_2$ are direct summand of free module.*

0.3 p-GROUPS

In this section, we introduce the concept of p-group, and finite group with a BN-pair. We also recall their properties, which we need in this dissertation. The basics concepts of group theory can be taken from the following book [13].

DEFINITION 0.3.1. *([13], p.93). Let G be a finite group, and let p be a prime natural number. Then G is called p-group if every element in G ($\forall g \in G$) has order a power of prime number as form: $p^r, r \in \mathbb{N}$, where $g^{p^r} = 1$, (i.e. $|g| = p^r$). Moreover, G is p-group if and only if $|G| = p^r$.*

DEFINITION 0.3.2. *([13], p.93). Let G be a p-group, and let U be a subgroup of group G. If U is a p-group, then U is p-subgroup of G; i.e. Each a subgroup U of p-group G is also p-group.*

The following definition shows the concept of the Sylow p-subgroup.

DEFINITION 0.3.3. *([13], p.95). Let G be a group, $|G| = p^r m$ where $gcd(p, m) = 1$, r is a non-negative integer ($p^r \mid |G|$). Then the subgroup of G of order p^r is called Sylow p-subgroup of G, and denote for all Sylow p-subgroup of G by $Syl_p(G)$.*

THEOREM 0.3.4. *(Third Sylow Theorem)([13], p.95). Let G be a finite group of order $n = p^r m$, where $gcd(p, m) = 1$, r is a non-negative integer. Then the number of Sylow p-subgroups of G divides $|G|$, and is of the form $\lambda p + 1$ for some $\lambda \geq 0$.*

EXAMPLE 0.3.5. *Let $G = S_3 =< a, b|a^3 = b^2 = 1, bab^{-1} = a^{-1} >$, the order of S_3 is 6. We will find the Sylow 2-subgroup of S_3 of order 2, and the Sylow 3-subgroup of S_3 of order 3.*

If $p = 3$, $\quad k = 1 + 3\lambda$.
At $\lambda = 0 \Rightarrow k = 1$, hence $1 \mid 6$, and at $\lambda = 1 \Rightarrow k = 4$, hence $4 \nmid 6$. Then the Sylow 3-subgroup of S_3 of order 3 is $A_3 = \{1, a, a^2\}$, where $A_3 =< a >$ is cyclic.

If $p = 2$, $\quad k = 1 + 2\lambda$.
At $\lambda = 0 \Rightarrow k = 1$, hence $1 \mid 6$, at $\lambda = 1 \Rightarrow k = 3$, hence $3 \mid 6$, and at $\lambda = 2 \Rightarrow k = 5$, hence $5 \nmid 6$. Then there are 3 Sylow 2-subgroups $< b >$, $< ab >$, and $< a^2 b >$, where they are cyclic.

If algebra A has a finite number of indecomposable A-modules, then A has a finite representation type as follows:

DEFINITION 0.3.6. *([11], p.73). Let A be a finite dimension F-algebra. Then algebra A has finite representation type if and only if A possesses a finite number of non-isomorphism classes of indecomposable A-modules.*

If the group G has a Sylow p-subgroup, where it is cyclic, then the group algebra FG has a finite number of non-isomorphism classes of indecomposable modules, according to the following well-known theorem.

THEOREM 0.3.7. *(G. Higman) ([3], p.64). Let F be a field of characteristic p, and let U be a Sylow p-subgroup of G. Then the group algebra FG has finite representation type if and only if a Sylow p-subgroup of G is cyclic.*

Proof. The proof can be found in ([3], Corollary 2.12.9). □

The following corollary shows that the trivial module is the only simple module in a p-group algebra.

COROLLARY 0.3.8. *([2], p.14). If F is a field of characteristic p, and G is a finite p-group, then the only simple FG-module is the trivial module.*

Proof. The identity element is the only one of order not divisible by p, so by theorem (0.1.23), FG has a unique simple module, namely the trivial module [2]. □

The following proposition shows that every projective FG-module has dimension divisible by the order of Sylow p-subgroup.

PROPOSITION 0.3.9. *([2], p.33). Let F be a field, let G be a group, let FG be a group algebra, and let U be a Sylow p-subgroup of G has order p^a, then every projective FG-module has dimension divisible by p^a.*

The following theorem shows that, for group G whose a Sylow p-subgroup is cyclic normal, the indecomposable FG-modules can be obtained as quotients of the projective indecomposable modules

THEOREM 0.3.10. *([6], p.478). Let D be a cyclic normal Sylow p-subgroup of a finitely group G, and let F be any field of characteristic prime number p, not necessarily a splitting field for G.*
Let $\{U_i : 1 \leq i \leq s\}$ be a basic set of projective indecomposable FG-module, and let

$$D =< x : x^{p^d} = 1 >, \ N = rad(FD) = (x-1)FD,$$

so N is nilpotent of exponent p^d. Put

$$M_{ij} = U_i/N^j U_i; \qquad 1 \leq j \leq p^d, 1 \leq i \leq s.$$

Then the $s.p^d$ modules $\{M_{ij}\}$ are a full set of non-isomorphic indecomposable FG-module.

From the previous theorem, we find that:

THEOREM 0.3.11. *([2], p.42) Let F be a field of characteristic p, let G be a finite group has a cyclic normal Sylow p-subgroup, and let FG be a group algebra. Then any indecomposable FG-module is a homomorphic image of the projective indecomposable module.*

The following definition introduces the concept of a finite group with BN-pairs.

DEFINITION 0.3.12. *([7],p.561). Let W be a finite group. If*

$$W =< s_1, s_2, \ldots, s_n : (s_i s_j)^{m_{ij}} = 1 \quad \forall i, j >,$$

where the $\{m_{ij}\}$ are positive integers such that

$$m_{ii} = 1, m_{ij} > 1 \text{ if } i \neq j, \text{ and } m_{ij} = m_{ji} \text{ for all } i, j.$$

Then W is called a finite Coxeter group. The pair (W, S) is called a finite Coxeter system, where $S = \{s_1, s_2, \ldots, s_n\}$ is a set of generators of W.

EXAMPLE 0.3.13. *The cyclic group of order 2; $W =< S_1 | S_1^2 = 1 >$, then W is finite Coxeter group.*

DEFINITION 0.3.14. *([7], p.576). Let G be a finite group, and let B, N be a pair of subgroups of G. Then a finite group with a BN-pair satisfying the following axioms:*

1. *$G =< B, N >$;*
2. *$B \cap N \trianglelefteq N$;*
3. *Let $W = N/B \cap N$, and for each $w \in W$ choose a coset representative $\dot{w} \in N$. Then W is generated by a set $S = \{s_1, s_2, \ldots, s_n\}$ such that*

$$\dot{s}_i B \dot{w} \subseteq B \dot{w} B \cup B \dot{s}_i w B$$

and

$$\dot{s}_i B \dot{s}_i \neq B,$$

for each $w \in W$ and each $s_i \in S$.

Details can be found in ([7], section 65).

0.4 CHAINS

In this section, we introduce the concept of composition series, in which we use it to determine all indecomposable A-modules.

DEFINITION 0.4.1. *([13], p.375). Let A be an F-algebra, and let D be an A-module. A normal series for D is a chain of A-submodules*

$$D = D_0 \supset D_1 \supset D_2 \supset ... \supset D_n$$

is called composition series for A-module D if D_i/D_{i+1} for all $0 \leq i < n$ is simple A-module, where D_i/D_{i+1} for all $0 \leq i < n$ is called composition factors.

EXAMPLE 0.4.2. *([10], p.64). Let A be an F-algebra, let $D = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$ be upper triangular matrices over a field F, and let $D = A$-module B,*

$$B_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in F \right\} \quad \& \quad B_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in F \right\}.$$

The chain

$$0 = B_0 \subset B_1 \subset B_2 \subset B_3 = B,$$

is composition series, because the dimension of composition factor B_i/B_{i-1} for all $i = 1, 2, 3$, is one. Since every simple module has dimension one as stated in ([10], Example 3.2, p.61), then the composition factor B_i/B_{i-1} is simple.

DEFINITION 0.4.3. *([9], p.255). Let A be a finite dimension F-algebra, and let D be an A-module. Then D satisfies the descending chain condition (DCC) if every descending chain of submodules is finite. Thus, D is Artinian, i.e.*

$$D_1 \supset D_2 \supset D_3 \supset \dots$$

of A-submodules of D, there is $r \in \mathbb{N}$ such that $D_i = D_r \; \forall i \geq r$.

DEFINITION 0.4.4. *([9], p.255). Let A be a finite dimension F-algebra, and let B be an A-module. Then B satisfies the ascending chain condition (ACC) if every ascending chain of submodules is finite. Thus, B is Noetherian, i.e.*

$$B_1 \subset B_2 \subset B_3 \subset \dots$$

of A-submodules of B, there is $m \in \mathbb{N}$ such that $B_i = B_m \; \forall i \geq m$.

Consequently, A is Artinian if the A-module A satisfies the DCC, or A is Noetherian if the A-module A satisfies the ACC.

THEOREM 0.4.5. *(Jordan-Holder Theorem) ([13], p.375). Let A be a finite dimension F-algebra. Then any two composition series of A are equivalent.*
Details can be found in [9, 13].

We will use the radical series in the determination of the indecomposable A-modules. So, we introduce the concept of the radical series.

DEFINITION 0.4.6. *([2], p.3). Let A be a finite dimension F-algebra. Then the radical of A is equal to each of the following;*

1. *$rad(A)$ is the intersection of all the maximal submodules of A;*

2. $rad(A)$ is the largest nilpotent ideal of algebra A.

REMARK. *([3], p.1). Let M be an A-module. Then $M/radM$ is the head of M.*

DEFINITION 0.4.7. *([3], p.1). Let A be a finite dimension F-algebra, and let M be an A-module. Then the socle of M is the sum of all the irreducible submodules of M, i.e.*

$$soc(M) = m_1 \oplus m_2 \oplus \ldots \oplus m_n,$$

where m_i $\forall i = 1, 2, \ldots, n$ is irreducible submodules of M.
Moreover, if $M = soc(M)$, then M is completely reducible (semisimple).

COROLLARY 0.4.8. *([2], p.3). Let A be a finite dimension F-algebra. If $radA = 0$, then A is semisimple.*

DEFINITION 0.4.9. *([11], p.129). Let A be an F-algebra, let M be an A-module. Then the radical series of M is*

$$M = rad^0(M) \supset rad^1(M) \supset rad^2(M) \supset \ldots \supset rad^r(M) = 0.$$

Also, the socle series of M is

$$0 = soc^0(M) \subset soc^1(M) \subset soc^2(M) \subset \ldots \subset soc^r(M) = M.$$

From ([18], section 8, p.25), there exists $r \in \mathbb{N}$ such that $rad^r(M) = 0$. By Loewy series, then

$$\begin{array}{c} M/rad(M) \\ rad(M)/rad^2(M) \\ rad^2(M)/rad^3(M) \\ \vdots \\ rad^{r-1}(M), \end{array}$$

where the head is $M/rad(M)$ and the socle is $rad^{r-1}(M)$. More details on Loewy series are given in [11, 18].

The following theorem shows that, the head of projective indecomposable FG-module is isomorphic to socle of M.

THEOREM 0.4.10. *([2], p.43). Let F be a field, let G be a group, and let FG be a group algebra, and let M be a projective indecomposable FG-module. Then $M/rad(M) \cong soc(M)$.*

The head of projective indecomposable FG-module is simple as follows:

COROLLARY 0.4.11. *([2], p.41). Let P be a projective indecomposable FG-module. Then $soc(P)$ is simple.*

The following definition shows the concept of uniserial module and uniserial algebra.

DEFINITION 0.4.12. *([7], p.505). Let A be a finite dimension F-algebra, and let M be a finite generated A-module. Then M is a uniserial module if M has a unique composition series. Hence, every submodule and factor module of M is then also uniserial. While A is called a uniserial F-algebra, if every projective indecomposable module is a uniserial module.*

Also, let N =$radA$, and let the radical series

$$M \supset NM \supset N^2M \supset \ldots \supset 0$$

be a chain of submodules such that each quotient $N^iM/N^{i+1}M$ is semisimple A-module. Then M is uniserial $\Leftrightarrow$ $N^iM/N^{i+1}M$ is simple for all $i = 0, 1, 2, \ldots$.

0.5 EXACT SEQUENCE

We benefit from this section in the determination of the constructing an indecomposable A-module.

We first introduce the definition of exact sequence.

DEFINITION 0.5.1. *([9], p.283). Let A be an F-algebra, and let $A_0, A_1, A_2, \ldots, A_n$ be an A-modules. The sequence*

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$

of A-module homomorphism f_i is exact at A_i if and only if

$$Im(f_i) = ker(f_{i+1}), \quad \forall i = 0, 1, ..., n.$$

A short exact sequence is an exact sequence, in which M is an A-module, and V_n submodule of M, then

$$0 \to V_n \xrightarrow{i} M \xrightarrow{f} V_m \to 0, \tag{1}$$

where $i : V_n \to M$ is inclusion map, and $f : M \to V_m$ is surjective map, also $Im(i) = ker(f)$.

REMARK. *Since V_n submodule of M, then*

$$0 \to V_n \xrightarrow{i} M \xrightarrow{f} M/V_n \to 0$$

is short exact, then from (1), $V_n \cong i(V_n)$ a submodule of M, and $V_m \cong M/i(V_n)$.

DEFINITION 0.5.2. *([9], p.283). Let A be an F-algebra, let*

$$0 \to V_n \xrightarrow{i} M^{\backprime} \xrightarrow{f} V_r \to 0$$

be a short exact sequence of an A-module, and let V_r, V_n be two submodules of A-module $M^{\backprime}$, where $M^{\backprime} = V_n \oplus V_r$. Then the short exact sequence $0 \to V_n \xrightarrow{i} V_n \oplus V_r \xrightarrow{f} V_r \to 0$ is called split exact sequence.

Here, we introduce the concept of non-split extension, and we show the structure of any indecomposable A-module M.

DEFINITION 0.5.3. *([6], p.175). The short exact sequence in definition (0.5.1) is called a non-split extension of V_m by V_n. We denote it by $Ext_A(V_m, V_n)$.*
i.e. $Ext_A(V_m, V_n) \neq 0$, then M is indecomposable A-module.

The structure of M is $\begin{array}{c} V_m \\ | \\ V_n \end{array}$ *. But, the short exact sequence in definition (0.5.2) is called a split extension.*
i.e., $Ext_A(V_r, V_n) = 0$, then $M^{\backprime}$ is decomposable A-module.

There are very important theorems to determine the structure of projective indecomposable A-module, which we need in chapter 5 they are:

THEOREM 0.5.4. *([2], p.76). There is a non-split short exact sequence of FG-modules,* $1 \leqslant i < p-1$,

$$0 \to V_{p-i-1} \to V \to V_i \to 0.$$

THEOREM 0.5.5. *([2], p.77). There is a non-split short exact sequence of FG-modules,* $1 < i \leqslant p-1$,

$$0 \to V_{p+1-i} \to V \to V_i \to 0.$$

THEOREM 0.5.6. *([2], p.47). Let V be an FG-module, and let P be a projective FG-module. Then $V \otimes P$ is also projective.*

THEOREM 0.5.7. *([2], p.50). If $2 \leqslant n < p$, then $V_2 \otimes V_n \cong V_{n-1} \oplus V_{n+1}$.*

0.6 CATEGORY

In this section, we describe a category $modA$. We also recall some of its properties, where $modA$ is the category of finite dimensional left A-modules.

DEFINITION 0.6.1. *([13], p.52). Let A be an algebra over the field F. Then a category is a class $modA$ of objects $D, B, ...$ together with*

1. *A class of disjoint sets, every element of $Hom_{modA}(D,B)$ is called a morphism from D to B ($\forall f \in Hom_{modA}(D,B)$; $f : D \longrightarrow B$ is A-map).*
2. *For all $D, B, C \in modA$, there exists a function*

   $$Hom_{modA}(D,B) \times Hom_{modA}(B,C) \longrightarrow Hom_{modA}(D,C),$$

   is called the composition by $(f,g) \longmapsto g \circ f$ such that $f \in Hom_{modA}(D,B)$ and $g \in Hom_{modA}(B,C)$; we have $g \circ f : D \longrightarrow C$, in which

   - *Associativity. $h \circ (g \circ f) = (h \circ g) \circ f$, where $f : D \to B$, $\quad g : B \to C$, and $h : C \to E$ are morphisms of $modA$.*
   - *The identity morphism on B, $1_B \in Hom_{modA}(B,B)$, where $1_B : B \longrightarrow B$, $f : D \to B$, and $g : B \to C$ such that*

     $$1_B \circ f = f, \text{ and } g \circ 1_B = g.$$

   See, examples in ([13], p.53).

DEFINITION 0.6.2. *([13], p.465). Let A be an algebra over the field F, and let $MmodA$ be a category whose objects are the covariant functors.*

$$K : modA \longrightarrow modF$$

is a pair of functions, where $modF$ is the category of vector spaces over F. The first, for each the objects of $modA$ X, then $K(X)$ is object of $modF$.
The second, for each a morphism, $t \in Hom_{modA}(X, X')$, where $X, X' \in modA$, then $K(t) \in Hom_{modF}(K(X), K(X'))$ in $modF$ such that:

- *$K(1_X) = 1_{K(X)}$ such that 1_X is identity morphism of $modA$.*
- *For all $f \in Hom_{modA}(D,B)$, and for all $g \in Hom_{modA}(B,C)$, then $K(g \circ f) = K(g) \circ K(f)$, in which $K(f) : K(D) \longrightarrow K(B)$ and $K(g) : K(B) \longrightarrow K(C)$, where composite $g \circ f$ is defined.*

  Details can be found in ([13], Section X.1).

COROLLARY 0.6.3. *([15], p.4). If $V, V' \in modA$, then $Mor_{MmodA}((\quad, V), (\quad, V')) \neq 0$ if and only if $(V, V') \neq 0$, where $Mor_{MmodA}((\quad, V), (\quad, V'))$ is denoted the space of morphisms from $(\quad, V)$ to $(\quad, V')$.*

Chapter 1

THE CONCEPT OF PSEUDOBLOCKS

In [16], the concept of the pseudoblocks of an endomorphism algebra of a module was introduced in terms of its indecomposable summands. Here, we borrow this notion and introduce it for any finite dimensional algebras.

1.1 THE PSEUDOBLOCK PRINCIPLE

Here, we present the concept of pseudoblocks of finite dimensional algebras, where we use the notation $\underset{PSA}{\approx}$ to mean "lie in the same pseudoblock of A" or "pseudoblock linkage principle".

DEFINITION 1.1.1. *([16], p.2366). Let A be a finite dimension algebra over the field F. Then the pseudoblock linkage principle $\underset{PSA}{\approx}$ on the class of all indecomposable A-module $IndA$ as follows: If $X, Y \in IndA$, then $X \underset{PSA}{\approx} Y$ if and only if there is a sequence of modules $X = X_1, X_2, ..., X_t = Y$ such that for all $i \in \{1, 2, ..., t\}$ either*

$$(X_i, X_{i+1})_A \neq 0 \quad or \quad (X_{i+1}, X_i)_A \neq 0.$$

It is clear that $\underset{PSA}{\approx}$ is an equivalence relation on $IndA$, and hence $IndA$ is partitioned into equivalence classes $IndA/ \underset{PSA}{\approx}$.

REMARK. *In [16], let A be a finite dimension F-algebra, and let Y be a finite dimension A-module*

$$Y = d_1Y_1 \oplus d_2Y_2 \oplus \ldots \oplus d_rY_r.$$

The notion of pseudoblocks of an endomorphism algebra $End_A(Y); Y$ in the category of finite dimensional left A-modules ($modA$) in terms of the indecomposable direct summands of the module Y, and it shows (Theorem (1.3.4)) that the pseudoblocks of $End_A(Y)$ control the (Brauer) linkage principle of the simple $End_A(Y)$-modules in the light of the Brauer-Fitting correspondence. We will explain this in detail in the section (1.3).

1.2 THE BRAUER LINKAGE PRINCIPLE

In this section, we explain the concept of the Brauer linkage principle $\underset{A}{\approx}$ on $modA$.

DEFINITION 1.2.1. *([3], p.13). Let F be a field of characteristic $p > 0$, and let A be an F-algebra. If we can write $1 = e_1 + e_2 + ... + e_r$, where 1 is the identity element in A, and e_i for all $i = 1, 2, ..., r$ are orthogonal central idempotents, then $A = B_1 \oplus B_2 \oplus ... \oplus B_r$, where $B_i = Ae_i$ for all $i = 1, 2, ..., r$ are indecomposable as two-sided ideals. Thus, B_i for all $i = 1, 2, ..., r$ are called the blocks of A.*
Moreover, e_i for all $i = 1, 2, ..., r$ are primitive if and only if B_i are indecomposable.

The direct summands of a module are unique up to isomorphism and order according to the following well-known theorem.

THEOREM 1.2.2. *(Krull-Schmidt Theorem) ([2], Theorem3). Let V be an A-module. If $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ and $V = U_1 \oplus U_2 \oplus \ldots \oplus U_r$ are two decompositions into the direct sum of indecomposable modules, then $r = s$ and after suitable renumbering, $U_i \cong V_i$ for all $i = 1, 2, \ldots, s$.*

Proof. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ and $V = U_1 \oplus U_2 \oplus \ldots \oplus U_r$. Then from definition (1.2.1), $V_i = Ve_i \quad \forall i = 1, 2, \ldots, s$ and $U_j = Vf_j \quad \forall j = 1, 2, \ldots, r$, where e_i, f_j for all i, j are primitive idempotents.
Let 1_V be the identity element of V; it has the decomposition $1_V = e_1 \oplus e_2 \oplus \ldots \oplus e_s$.
From theorem(0.2.8), $1_V = e_1 \oplus e_2 \oplus \ldots \oplus e_s$ is unique; i.e. $1_V = e_1 \oplus e_2 \oplus \ldots \oplus e_s = f_1 \oplus f_2 \oplus \ldots \oplus f_r$, where $s = r$, hence $e_i = f_i \quad \forall 1 \leq i \leq r$.
Then,

$$\bigoplus_{i=1}^{r} Ve_i = \bigoplus_{i=1}^{r} Vf_i.$$

Thus, $V_i \cong U_i$ for all $i = 1, 2, \ldots, r$.

□

The following lemma shows that, the blocks of algebra A are unique.

LEMMA 1.2.3. *([3], Lemma1.6.1). Let A be an F-algebra. If $A = B_1 \oplus B_2 \oplus \ldots \oplus B_r$, where $B_i = Ae_i$, e_i is central primitive orthogonal idempotent for all $i = 1, 2, \ldots, r$, then the decomposition of A is unique, i.e. the blocks $B_i \quad \forall i = 1, 2, \ldots, r$. are unique.*

Proof. From Krull-Schmidt Theorem (1.2.2), the decomposition of A is unique, i.e. the blocks B_i for all $i = 1, 2, \ldots, r$ are unique. □

REMARK. *([2], p.93).*

1. *We can distribute an indecomposable A-module M lies in the block B_i ($M \in B_i$) if $B_iM = M$ and $B_jM = 0$, where $B_i = Ae_i$ for all $j, j \neq i$.*

2. *If a module belongs to a block, then all of its composition factors belong to that block.*

We can classify all indecomposable A-modules in blocks by the following theorem:

THEOREM 1.2.4. *([2], Section 13). Let A be an F-algebra, and let X, Y be two simple A-modules. Then X, Y lie in the same block (i.e. $X \underset{A}{\approx} Y$) if and only if there is a sequence from projective indecomposable modules $P_j = P_1, P_2, \ldots, P_t = P_k$ corresponding the simple A-modules such that for all $i \in \{1, 2, \ldots, t\}$ either*

$$(P_i, P_{i+1})_A \neq 0 \quad \text{or} \quad (P_{i+1}, P_i)_A \neq 0.$$

Moreover, If X, Y are two indecomposable A-modules with there exists A-module homomorphism from A-module X to A-module Y ($(X, Y)_A \neq 0$), then $X \underset{A}{\approx} Y$.

More details on block theory are given in [2], [11].

1.3 CONNECTION WITH THE BLOCKS OF ENDOMORPHISM ALGEBRAS

In this section, we study compatibility between the pseudoblocks of indecomposable direct summands of the module Y and the Brauer linkage principle of the simple $End_A(Y)$-modules by Brauer-Fitting correspondence. Then we introduce the notion of pseudoblocks of indecomposable direct summands of the module Y for any finite dimensional algebra.

DEFINITION 1.3.1. *([17], p.897). Let A be a finite dimension F-algebra, where F is an algebraically closed field of characteristic $p > 0$, let Y be a finite dimension A-module and*

$$Y = d_1Y_1 \oplus d_2Y_2 \oplus \ldots \oplus d_rY_r,$$

where Y_i is indecomposable A-module, and let $E(Y) = End_A(Y)$ be an endomorphism. Then the Brauer-Fitting correspondence is a bijection between the set $Irr(E(Y))$ of isomorphism classes of simple $E(Y)$-module and the set $Inds(Y)$ of isomorphism classes of indecomposable direct A-summands of Y, where $Inds(Y) = \{Y_1, Y_2, \ldots, Y_r\}$. Then the Brauer-Fitting correspondence is given by the following bijection

$$Y_i \longleftrightarrow S_i = \frac{P_i}{radP_i}, \tag{1.1}$$

where $P_i = (Y_i, Y)_A$ is projective indecomposable $E(Y)$-module, $radP_i$ is the radical of P_i, and $dimS_i = d_i$ represents the number of times Y_i appears for all $1 \leq i \leq r$.

REMARK. *Connection between the representation theory of the endomorphism algebra $E(Y)$ and the representation theory of A in definition (1.3.1) is provided by Brauer-Fitting Correspondence.*

THEOREM 1.3.2. *([15], p.4). If $S_i \underset{E(Y)}{\approx} S_j$, then $Y_i \underset{A}{\approx} Y_j$.*

Proof. Let $S_i \underset{E(Y)}{\approx} S_j$. By theorem (1.2.4), there exists a sequence of projective indecomposable modules

$$P_i = P_1, P_2, \ldots, P_t = P_j,$$

corresponding the simple E(Y)-modules such that for all $r \in \{1, 2, \ldots, t\}$ either

$$(P_r, P_{r+1})_{E(Y)} \neq 0 \quad \text{ or } \quad (P_{r+1}, P_r)_{E(Y)} \neq 0, \tag{1.2}$$

but $P_r = (Y_r, Y)_A = Hom_A(Y_r, Y)$, hence from (1.2),
$Hom_{E(Y)}((Y_r, Y), (Y_{r+1}, Y) \neq 0$ or $Hom_{E(Y)}((Y_{r+1}, Y), (Y_r, Y) \neq 0 \quad \forall 1 \leqslant r \leqslant t$.
Then the space of morphisms $Mor_{_{MmodA}}((Y_r, -), (Y_{r+1}, -)) \neq 0$ or
$Mor_{_{MmodA}}((Y_{r+1}, -), (Y_r, -)) \neq 0 \quad \forall 1 \leqslant r \leqslant t$.
From corollary (0.6.3), $Mor_{_{MmodA}}((Y_r, -), (Y_{r+1}, -)) \neq 0$ if and only if $(Y_r, Y_{r+1})_A \neq 0$, and
$Mor_{_{MmodA}}((Y_{r+1}, -), (Y_r, -)) \neq 0$ if and only if $(Y_{r+1}, Y_r)_A \neq 0$. Hence from theorem (1.2.4),

$$(Y_r, Y_{r+1})_A \neq 0 \text{ or } (Y_{r+1}, Y_r)_A \neq 0 \quad \forall 1 \leqslant r \leqslant t.$$

Thus, $Y_i \underset{A}{\approx} Y_j$. □

But the converse of the theorem (1.3.2) is not true by providing a counterexample.

EXAMPLE 1.3.3. *([15], p.6). Let F be a field of characteristic 2, and let $G = SL(2,4) \cong A_5$.*
From theorem(0.1.23), A_5 has four 2-regular conjugacy classes $\{(1),(123),(12345),(12345)^2\}$. Hence, $A_5 \cong SL(2,4)$ has 4 non-isomorphic classes of irreducible representations; namely $1, 2_1, 2_2$, and 4, where 1 is a trivial module and 4 is a Steinberg module (St_G). Then there are four projective indecomposable modules.
Since Klein four $V = V_4$ is subgroup of A_5, then V is the Sylow 2-subgroup of G, where $V \cong C_2 \times C_2$ is not cyclic (i.e. FG is infinite representations type).
Since $|V| = 4$, then the dimension of the Steinberg module is four.

Let $Y = ind_V^G(1) = FG \otimes_{FV} 1$. From theorem(0.1.14), then

$$dim_F Y = |G:V|dim(1) = |G|/|V| = 60/4 = 15.$$

From (Krull-Schmidt Theorem (1.2.2)) then,

$$Y = 1 \oplus \begin{matrix} 2_1 \\ 1 \\ 2_2 \end{matrix} \oplus \begin{matrix} 2_2 \\ 1 \\ 2_1 \end{matrix} \oplus 4. \tag{1.3}$$

FG has two blocks. $1, 2_1, 2_2$, and 4 are irreducible FG-modules, of which the first three irreducible FG-modules belong to the same block, and the last one is the Steinberg module 4.

Y is multiplicity free FG-module. Hence, from Brauer-Fitting correspondence in definition (1.3.1), there exists one-to-one correspondence between indecomposable projective FG-module $\{1, \begin{matrix} 2_1 \\ 1 \\ 2_2 \end{matrix}, \begin{matrix} 2_2 \\ 1 \\ 2_1 \end{matrix}, 4\}$ and the simple $End_{FG}(Y)$-module $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ respectively, where $dim\psi_i = 1 \quad \forall i = \{1,2,3,4\}$.
The group G has a split BN-pair, in which $B = \left\{ \begin{pmatrix} x & \lambda \\ 0 & x^{-1} \end{pmatrix} : x \in F_4^*, \lambda \in F_4 \right\}$, $N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} : x, y, z \in F_4^* \right\}$, and $H = B \cap N = \left\{ h(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in F_4^* \right\}$, where $H \trianglelefteq N$ and $N/H = W =< w/w^2 = 1 >$ is Coxeter group.
The set of multiplicative F-characters of H is $\hat{H} = \{\chi_r : r = 0,1,2\}$, where $\chi_r(h(x)) = x^r \quad \forall x \in F_4^*$.
Hence $P(\chi_r) = \{w_i \in R : \chi_r|_{V \cap {}^{w_i}V} = 1\}$, $R = \{w\}$ is the set of simple generators, then $P(\chi_0) = R = \{w\}$, and $P(\chi_1) = P(\chi_2) = \phi$.
From equation (1.3), then

$$Y = Y(\chi_0, R) \oplus Y(\chi_1, \phi) \oplus Y(\chi_2, \phi) \oplus Y(\chi_0, \phi),$$

this means that the head $M(\chi_0, R) = 1, M(\chi_1, \phi) = 2_1, M(\chi_2, \phi) = 2_2$, and $M(\chi_0, \phi) = 4$, where $M(\chi_r, P(\chi_r))$ is composition factor of $Y(\chi_r, P(\chi_r))$.

The simple $End_{FG}(Y)$-module is $\psi_1 = \psi(\chi_0, R), \psi_2 = \psi(\chi_1, \phi), \psi_3 = \psi(\chi_2, \phi)$, and $\psi_4 = \psi(\chi_0, \phi)$.
From theorem 2 in paper ([15], p.5), then $End_{FG}(Y)$ has three blocks as follows: $\{\psi_1\}$, $\{\psi_2, \psi_3\}$ and $\{\psi_4\}$. Then $1 \underset{FG}{\approx} \begin{matrix} 2_1 \\ 1 \\ 2_2 \end{matrix}$, but $\psi_1 \underset{E(Y)}{\not\approx} \psi_2$.

Thus, if $Y_i \underset{A}{\approx} Y_j$, then $S_i \underset{E(Y)}{\not\approx} S_j$.

So from theorem (1.3.2), and example (1.3.3), the Brauer-Fitting correspondence is not compatible with the (Brauer) block for the indecomposable A-summands of Y and the (Brauer) block for the simple $End_A(Y)$-modules.

Moreover, the concept of the pseudoblocks for the indecomposable A-summand of Y is compatible with the block for the simple $End_A(Y)$-module by the Brauer-Fitting correspondence. The follows theorem shows it.

THEOREM 1.3.4. *([16], p.2367). $S_i \underset{E(Y)}{\approx} S_j$ if and only if $Y_i \underset{PSA}{\approx} Y_j$.*

Proof. From theorem (1.3.2), if $S_i \underset{E(Y)}{\approx} S_j$, then $Y_i \underset{A}{\approx} Y_j$. Since

$$(Y_r, Y_{r+1})_A \neq 0 \quad \text{or} \quad (Y_{r+1}, Y_r)_A \neq 0 \quad \forall 1 \leqslant r \leqslant t,$$

then $Y_i \underset{PSA}{\approx} Y_j$.

Conversely, if $Y_i \underset{PSA}{\approx} Y_j$, then

$$(Y_r, Y_{r+1})_A \neq 0 \quad \text{or} \quad (Y_{r+1}, Y_r)_A \neq 0 \quad \forall 1 \leqslant r \leqslant t.$$

From corollary (0.6.3), if and only if $Mor_{_{MmodA}}((Y_r, -), (Y_{r+1}, -)) \neq 0$.
Then $Hom_{End(Y)}((Y_r, Y), (Y_{r+1}, Y)) \neq 0$. Since $P_r = (Y_r, Y)_A$ and $P_{r+1} = (Y_{r+1}, Y)_A$, then $(P_r, P_{r+1})_{E(Y)} \neq 0$, similarly $(P_{r+1}, P_r)_{E(Y)} \neq 0$.
Hence, every sequence of projective indecomposable modules

$$P_i = P_1, P_2, \ldots, P_t = P_j,$$

is

$$(P_r, P_{r+1})_{E(Y)} \neq 0 \quad \text{or} \quad (P_{r+1}, P_r)_{E(Y)} \neq 0 \qquad \forall 1 \leq r \leq t.$$

Since P_i, P_j are the projectives cover of two simple $End_A(Y)$-modules S_i, S_j, then from theorem (1.2.4), $S_i \underset{E(Y)}{\approx} S_j$. □

REMARK. *The pseudoblock of finite dimensional algebras control the Brauer linkage principle of simple $End_A(Y)$-modules by the Brauer-Fitting correspondence.*

1.4 A USEFUL CRITERION

In this section, we prove a criterion (lemma 1.4.1) for having nonzero homomorphism space between two indecomposable modules. This criteria simplifies the determination of the pseudoblock linkage principle by looking at the composition series of modules.

As the pseudoblock linkage principle is defined in terms of a sequence of module homomorphisms, the following lemma describes a criterion, which simplifies the relation $\underset{PSA}{\approx}$.

LEMMA 1.4.1. *Let $X, Y \in IndA$. Then $(X, Y)_A \neq 0$ if and only if $\exists K \leqslant_A X$: $X/K \cong$ a submodule of Y.*

Proof. Let $(X, Y)_A \neq 0$. If $0 \neq f \in (X, Y)_A$, where $f : X \longrightarrow Y$ is an A-module homomorphism.
$K = kerf \leqslant X$, and $X/K \cong Imf \leqslant Y$ from first isomorphism theorem.

Conversely, if there exists $K \leqslant X$ such that $X/K \cong T \leqslant Y$, then $g : X/K \longrightarrow Y$ and natural map $\theta : X \longrightarrow X/K$.
Hence, the composing $f = g \circ \theta : X \longrightarrow Y$, then $(X, Y)_A \neq 0$. $\square$

This criterion is illustrated by the following picture.

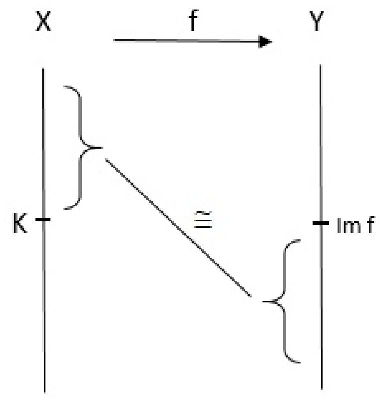

Figure 1.1:

Chapter 2

CONNECTION WITH THE BLOCK THEORY

In this chapter, we study the connection between the (Brauer) linkage principle and the pseudoblock linkage principle on $IndA$. We shall prove that the pseudoblock linkage principle implies the Brauer block linkage principle. Most results of this chapter are taken from [15], [16].

2.1 THE PSEUDOBLOCK PRINCIPLE IMPLIES THE LINKAGE PRINCIPLE

The following theorem demonstrates the connection between the two equivalence relations $\underset{PSA}{\approx}$, $\underset{A}{\approx}$.

THEOREM 2.1.1. *Every pseudoblock linkage principle implies the Brauer linkage principle, i.e. if $X, Y \in IndA$, then*

$$X \underset{PSA}{\approx} Y \Rightarrow X \underset{A}{\approx} Y.$$

Proof. If $X \underset{PSA}{\approx} Y$, then from definition(1.1.1), there is a sequence of modules $X = X_1, X_2, \ldots, X_t = Y$ in $IndA$ such that for all $i \in \{1, 2, \ldots, t\}$, either

$$(X_i, X_{i+1})_A \neq 0 \text{ or } (X_{i+1}, X_i)_A \neq 0,$$

from theorem (1.2.4), then $X_i \underset{A}{\approx} X_{i+1}$ or $X_{i+1} \underset{A}{\approx} X_i$ for all $i \in \{1, 2, \ldots, t\}$. Then all indecomposable A-modules $X = X_1, X_2, \ldots, X_t = Y$ lie in the same block. Hence, $X \underset{A}{\approx} Y$.

Otherwise, let $M_1, M_2 \in IndA$. If M_1 and M_2 lie in the blocks B_1 and B_2 respectively. Then $(M_1, M_2)_A = 0$ and $(M_2, M_1)_A = 0$; i.e. we do not have a sequence of modules $M_1 = N_1, N_2, \ldots, N_t = M_2$ in $IndA$, where for all $i \in \{1, 2, \ldots, t\}$

$$(N_i, N_{i+1})_A \neq 0 \text{ or } (N_{i+1}, N_i)_A \neq 0.$$

Then $M_1 \underset{PSA}{\not\approx} M_2$.

□

REMARK. *It follows from theorem (2.1.1) that the pseudoblock linkage principle $\underset{PSA}{\approx}$ is stronger than the Brauer block linkage principle $\underset{A}{\approx}$.*

2.2 AN ALGEBRA FOR WHICH THE TWO CONCEPTS DIFFER

In this section, we find that the pseudoblock linkage principle is different from the Brauer linkage principle, which comes from the converse of the theorem (2.1.1). We find in the example(2.2.1) that the converse of the theorem (2.1.1) is not true; i.e.

$$X \underset{A}{\approx} Y \nRightarrow X \underset{PSA}{\approx} Y.$$

EXAMPLE 2.2.1. *In example (1.3.3), FG in characteristic 2 has two blocks. $1, 2_1, 2_2$ and 4 are irreducible FG-modules, of which the first three irreducible FG-modules belong to the same block and the last one is the Steinberg module 4.*

Then, $IndsFG = \{1, \begin{matrix}2_1\\1\\2_2\end{matrix}, \begin{matrix}2_2\\1\\2_1\end{matrix}, 4\}$, *in which there are two blocks,* $B_1 = \{1, \begin{matrix}2_1\\1\\2_2\end{matrix}, \begin{matrix}2_2\\1\\2_1\end{matrix}\}$ *and* $B_2 = \{4\}$, *but the pseudoblocks of* $\Lambda = FG$ *are* $\{1\}$, $\{\begin{matrix}2_1\\1\\2_2\end{matrix}, \begin{matrix}2_2\\1\\2_1\end{matrix}\}$, *and* $\{4\}$.

We find that the two indecomposable modules $1, \begin{matrix}2_1\\1\\2_2\end{matrix}$ *belong to different pseudoblocks of* $\Lambda = FG$, *because there is no nonzero homomorphism from* 1 *to* $\begin{matrix}2_1\\1\\2_2\end{matrix}$ *and from* $\begin{matrix}2_1\\1\\2_2\end{matrix}$ *to* 1*; i.e.* $(\begin{matrix}2_1\\1\\2_2\end{matrix}, 1)_\Lambda = 0$ *&* $(1, \begin{matrix}2_1\\1\\2_2\end{matrix})_\Lambda = 0$. *Hence,*

$$1 \underset{\Lambda}{\approx} \begin{matrix}2_1\\1\\2_2\end{matrix}, \text{ but } 1 \underset{PS\Lambda}{\not\approx} \begin{matrix}2_1\\1\\2_2\end{matrix}.$$

REMARK. *In the previous example, it follows that in general some (Brauer) blocks of A will split into a union of pseudoblocks, and so, we have*

$$|IndA/ \underset{A}{\approx}| \leqslant |IndA/ \underset{PSA}{\approx}|.$$

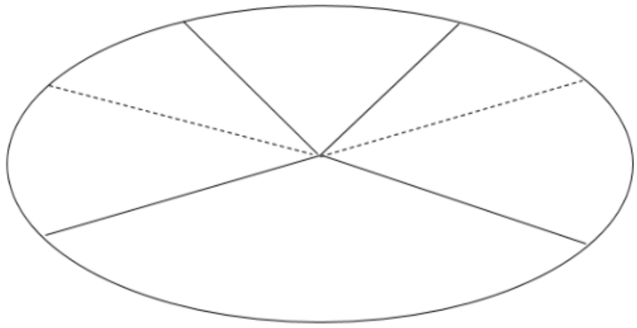

Figure 2.1: SOME BLOCKS IN *IndA* SPLIT INTO UNION OF PSEUDOBLOCKS

Chapter 3

CONNECTION WITH THE TENSOR PRODUCT

In this chapter, we revise the concepts of tensor product of algebras and modules. We also prove that the notion of pseudoblocks is compatible with the tensor product.

3.1 TENSOR PRODUCT OF ALGEBRAS AND MODULES

In this section, we revise the concept of tensor product of algebras and modules. The results of this section can be found in several standard algebra text books such as [13].

TENSOR PRODUCT OF MODULES

Let R be a ring, let M be a left R-module (${}_RM$), and let L be a right R-module (L_R). The tensor product of L and M over R is an Abelian group.

DEFINITION 3.1.1. *([6], p.23). Let R be a ring, let L_R and ${}_RM$ be right and left R-module, and let G be an Abelian group (Additive). Then balanced map (middle linear map) from $L\times M$ to G is the function $f : L\times M \longrightarrow G$ such that $\forall l, l_1, l_2 \in L$, $\forall m, m_1, m_2 \in M$, and $\forall a \in R$, where*

1. $f(l_1 + l_2, m) = f(l_1, m) + f(l_2, m)$;
2. $f(l, m_1 + m_2) = f(l, m_1) + f(l, m_2)$;
3. $f(la, m) = f(l, am)$.

DEFINITION 3.1.2. *([6], p.23). Let R be a ring, let M be a left R-module, and let L be a right R-module. Consider the Cartesian product $L \times M = \{(l, m) : l \in L, m \in M\}$. Let $(H, +)$ be the free Abelian group, which is generated by $L \times M$.*
$H =< L \times M >= \{\sum_{i=1}^{m} n_i(l, m) : l \in L, m \in M, n_i \in \mathbb{Z}\}$,
$H =< L \times M >= \mathbb{Z}(L \times M)$.

Let K be the subgroup of H, which is generated by

$$K =< (l_1+l_2, m)-(l_1, m)-(l_2, m), \quad (l, m_1+m_2)-(l, m_1)-(l, m_2), \quad (l, am)-(la, m) >$$

$\forall l, l_1, l_2 \in L$, $\forall m, m_1, m_2 \in M$, *and* $\forall a \in R$.
The tensor product of L and M over R is the quotient H/K (i.e. the tensor product

$L \underset{R}{\otimes} M$ *is additive abelian group).*

$$L \underset{R}{\otimes} M = H/K = \{t + K : t \in H\} = \{(l, m) + K : l \in L, m \in M\}.$$

REMARK. *([13], p.208).*

1. *The tensor product* $L \underset{R}{\otimes} M$ *is Abelian group;*
2. *The tensor product* $L \underset{R}{\otimes} M$ *is quotient group;*
3. *Since* $H/K \equiv L \underset{R}{\otimes} M$*, then* $(l, m) + K = l \underset{R}{\otimes} m$*, where*
   $(l, m) \in K \Leftrightarrow (l, m) + K = K \Leftrightarrow l \underset{R}{\otimes} m = 0_{L \underset{R}{\otimes} M}$*,*
   i.e. $H/K = L \underset{R}{\otimes} M$ *is generated by all elements (cosets) of the form* $(l \underset{R}{\otimes} m)$*.*
   So, (l, m) *has one of the following form*
   $$(l_1 + l_2, m) - (l_1, m) - (l_2, m) \quad or$$
   $$(l, m_1 + m_2) - (l, m_1) - (l, m_2) \quad or$$
   $$(la, m) - (l, am);$$
   $\forall l, l_1, l_2 \in L$*,* $\forall m, m_1, m_2 \in M$*, and* $\forall a \in R$*.*
4. *The zero of the tensor* $(0_L, 0_M) + K = 0_{L \underset{R}{\otimes} M}$ *is the zero element of the tensor.*
5. *We may have* $l \underset{R}{\otimes} m = l' \underset{R}{\otimes} m'$*, but* $l \neq l'$ *and* $m \neq m'$*, becaus*
   $$\begin{aligned} &(l, m) + K = (l', m') + K; \\ &(l, m) - (l', m') + K = K; \\ &so, \quad (l, m) - (l', m') \in K. \end{aligned}$$
6. *It is possible that* $L \underset{R}{\otimes} M = 0$*, but* $L \neq 0$ *&* $M \neq 0$*.*

   Some computation in tensor product.

   (i) $(l_1 + l_2) \underset{R}{\otimes} m = (l_1 \underset{R}{\otimes} m) + (l_2 \underset{R}{\otimes} m)$*;*

   (ii) $l \underset{R}{\otimes} (m_1 + m_2) = (l \underset{R}{\otimes} m_1) + (l \underset{R}{\otimes} m_2)$*;*

   (iii) $la \underset{R}{\otimes} m = l \underset{R}{\otimes} am = a(l \underset{R}{\otimes} m)$*;*

   (iv) $l \underset{R}{\otimes} 0_M = 0_L \underset{R}{\otimes} m = 0_L \underset{R}{\otimes} 0_M = 0_{L \underset{R}{\otimes} M}$*.*
7. *Let* L_R *and* ${}_RM$ *be two modules over a ring* R*, and let* $i : L \times M \longrightarrow L \underset{R}{\otimes} M$ *given by* $(l, m) \longmapsto l \underset{R}{\otimes} m$ *be a middle linear map. Then the map* i *is canonical middle linear map.*
8. *The typical element of* H *is a sum* $\sum_{i=1}^{r} n_i(l_i, m_i)$*, where* $n_i \in \mathbb{Z}$*,* $l_i \in L$*, and* $m_i \in M$*. Hence, the coset* $L \underset{R}{\otimes} M = H/K$ *is of the form* $\sum_{i=1}^{r} n_i(l_i \underset{R}{\otimes} m_i)$*.*

The following theorem shows that, the tensor product of two modules is a module.

THEOREM 3.1.3. *([5], p.70). Let G be a group, and let W, V be two vector spaces over the field F. If W, V are two FG-modules, then $V \underset{F}{\otimes} W$ is an FG-module.*

Proof. Let V, W are two FG-modules. From defintion (0.1.6), we have the following:

(i) $vx \in V$;

(ii) $(hv_1 + kv_2)x = h(v_1x) + k(v_2x), \qquad (v_1, v_2 \in V,\ x \in G,\ h, k \in F\ )$;

(iii) $v(xy) = (vx)y, \qquad (y \in G)$;

(iv) $v1 = v$.

Similarly; W is an FG-module, because

(i) $wy \in W$;

(ii) $(hw_1 + kw_2)y = h(w_1y) + k(w_2y), \qquad (w_1, w_2 \in W,\ y \in G,\ h, k \in F\ )$;

(iii) $w(yx) = (wy)x, \qquad (y, x \in G)$;

(iv) $w1 = w$.

Then, we prove that $V \underset{F}{\otimes} W$ is an FG-module;

1. $vx \underset{F}{\otimes} wy = (v \underset{F}{\otimes} w)xy \in V \underset{F}{\otimes} W$;
2. $$\begin{aligned}[h(v_1 \underset{F}{\otimes} w_1) + k(v_2 \underset{F}{\otimes} w_2)]xy &= h(v_1 \underset{F}{\otimes} w_1)xy + k(v_2 \underset{F}{\otimes} w_2)xy \\ &= h(v_1x \underset{F}{\otimes} w_1y) + k(v_2x \underset{F}{\otimes} w_2y);\end{aligned}$$
3. $(v \underset{F}{\otimes} w)(xy)z = (vx \underset{F}{\otimes} wy)z$;
4. $(v \underset{F}{\otimes} w)1 = v \underset{F}{\otimes} w$;

for all $w, w_1, w_2 \in W$, $v, v_1, v_2 \in V$, $x, y, z, xy \in G$ and $h, k \in F$.

Then, $V \underset{F}{\otimes} W$ is an FG-module. □

EXAMPLE 3.1.4. *Let $\mathbb{Z}$ be a ring, let $(\mathbb{Z}_n, \oplus_n)$ be a right $\mathbb{Z}$-module, and let $\mathbb{Q}$ be a left $\mathbb{Z}$-module. Then, $\mathbb{Z}_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \bar{0}$, $\forall n$. Let $m \in \mathbb{Z}_n$, and let $\dfrac{a}{q} \in \mathbb{Q}$. Then*

$$\begin{aligned} m \underset{\mathbb{Z}}{\otimes} \frac{a}{q} &= m \underset{\mathbb{Z}}{\otimes} n\frac{a}{nq}, \qquad (n \in \mathbb{Z}) \\ &= mn \underset{\mathbb{Z}}{\otimes} \frac{a}{qn} \\ &= \bar{0} \underset{\mathbb{Z}}{\otimes} \frac{a}{qn} = \bar{0}. \end{aligned}$$

DEFINITION 3.1.5. *([13], p.209). Let R be a Ring, and let $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ be two R-module homomorphism (R-map), where A, A' are two left R-module, and B, B' are two right R-module. Then there is unique R-module homomorphism $f \underset{R}{\otimes} g : A \underset{R}{\otimes} B \longrightarrow A' \underset{R}{\otimes} B'$ by $f \underset{R}{\otimes} g(a \underset{R}{\otimes} b) = f(a) \underset{R}{\otimes} g(b)$.*

TENSOR PRODUCT OF ALGEBRAS

Let F be a field, let X, Y be two F-algebras. We will write $X \underset{F}{\otimes} Y$ as follows $X \otimes Y$.

DEFINITION 3.1.6. *([19], p.85). Let V, W be two vector spaces over the same field F, and let $v_i \in V$, $w_j \in W$. Then the tensor product of two vector spaces is $V \underset{F}{\otimes} W = \sum_{i,j} \alpha_{ij}(v_i \underset{F}{\otimes} w_j)$, where $\alpha_{ij} \in F$.*

The following theorem shows that, the tensor product of two algebras is algebra.

THEOREM 3.1.7. *([5], p.72). Let F be a field, and let A_1, A_2 be two finite dimensional algebras over a field F (i.e. A_1, A_2 are two F-algebras). Then $A_1 \otimes A_2$ is an F-algebra.*

Proof. The proof can be found in ([5], p.72). □

COROLLARY 3.1.8. *([17], p.897). Let F be a field, let A_1, A_2 be F-algebras, let Y_1 be an A_1-module, and let Y_2 be an A_2-module. Then $Y_1 \otimes Y_2$ is an $A_1 \otimes A_2$-module, in which*

$$(a \otimes b)(x \otimes y) = ax \otimes by,$$

where for all $a \in A_1, b \in A_2, x \in Y_1,$ and $y \in Y_2$.

REMARK. *The radical of algebras satisfies*

$$rad(Y_1 \otimes Y_2) = rad(Y_1) \otimes Y_2 + Y_1 \otimes rad(Y_2). \tag{3.1}$$

3.2 COMPATIBLITY WITH THE TENSOR PRODUCT

Let F be a field, and let A_1, A_2 be two finite dimensional F-algebras.

THEOREM 3.2.1. *([17], p.897). Let Y_i be an A_i-module, where $i = 1, 2$. Then $E(Y_1 \otimes Y_2) \cong E(Y_1) \otimes E(Y_2)$ as F-algebras.*

LEMMA 3.2.2. *([17], p.897). Let A_1, A_2 be two finite dimensional F-algebras, and let X_i, Y_i be two A_i-modules, where $i = 1, 2$. Then,*

1. $(X_1, Y_1)_{A_1} \otimes (X_2, Y_2)_{A_2} \cong (X_1 \otimes X_2, Y_1 \otimes Y_2)_{A_1 \otimes A_2}$;
2. $rad((X_1 \otimes X_2, Y_1 \otimes Y_2)_{A_1 \otimes A_2}) = [rad((X_1, Y_1)_{A_1}) \otimes (X_2, Y_2)_{A_2}] \oplus [(X_1, Y_1)_{A_1} \otimes rad((X_2, Y_2)_{A_2})]$.

Proof. 1. The map $(f, g) \longmapsto f \otimes g$ is the balanced map, i.e. $(X_1, Y_1)_{A_1} \times (X_2, Y_2)_{A_2} \longrightarrow (X_1, Y_1)_{A_1} \otimes (X_2, Y_2)_{A_2}$.
Then, $(X_1 \times X_2, Y_1 \times Y_2)_{A_1 \times A_2} \longrightarrow (X_1, Y_1)_{A_1} \otimes (X_2, Y_2)_{A_2}$. From theorem (3.2.1), then $(X_1, Y_1)_{A_1} \otimes (X_2, Y_2)_{A_2} \cong (X_1 \otimes X_2, Y_1 \otimes Y_2)_{A_1 \otimes A_2}$.

2. From equation (3.1), $\big((radX_1 \otimes X_2) \oplus (X_1 \otimes radX_2), (radY_1 \otimes Y_2) \oplus (Y_1 \otimes radY_2)\big)_{A_1 \otimes A_2} = [rad\big((X_1, Y_1)_{A_1}\big) \otimes (X_2, Y_2)_{A_2}] \oplus [(X_1, Y_1)_{A_1} \otimes rad\big((X_2, Y_2)_{A_2}\big)]$.

□

The tensor product of two indecomposable modules is an indecomposable module according to the following theorem.

THEOREM 3.2.3. *Let F be a field, let A_1, A_2 be two F-algebras, let M be an indecomposable A_1-module, and let N be an indecomposable A_2-module. Then $M \otimes N$ is an indecomposable $A_1 \otimes A_2$-module.*

Proof. Let A_1, A_2 be two F-algebras, let M be an indecomposable A_1-module, and let N be an indecomposable A_2-module. From theorem (0.2.6), $End(M)$ and $End(N)$ are local. Hence, from lemma (0.2.5), the only idempotents in $End(M)$ and $End(N)$ are zero and the identity 1.
Since from theorem (3.2.1), $End_{A_1 \otimes A_2}(M \otimes N) \cong End_{A_1}(M) \otimes End_{A_2}(N)$,
then $End_{A_1 \otimes A_2}(M \otimes N)$ has idempotents are zero and 1. Hence, $End_{A_1 \otimes A_2}(M \otimes N)$ is local. Thus, $M \otimes N$ is an indecomposable $A_1 \otimes A_2$-module. □

The tensor product of two projective indecomposable modules is a projective indecomposable module from the following lemma.

LEMMA 3.2.4. *([17], p.898). Let P be a projective indecomposable A_1-module, and let Q be a projective indecomposable A_2-module. Then $P \otimes Q$ is a projective indecomposable $A_1 \otimes A_2$-module.*

Proof. Let P be a projective indecomposable A_1-module, and let Q be a projective indecomposable A_2-module.
From theorem (3.2.3), $P \otimes Q$ is an indecomposable $A_1 \otimes A_2$-module.

Let $P = A_1\alpha$ and $Q = A_2\beta$, where α and β are primitive idempotents

$$P \otimes Q = A_1\alpha \otimes A_2\beta = (A_1 \otimes A_2)(\alpha \otimes \beta).$$

Since α, β are primitive idempotents, i.e. $\alpha^2 = \alpha$ and $\beta^2 = \beta$, also $\alpha \neq \alpha_1 + \alpha_2$ and $\beta \neq \beta_1 + \beta_2$, where $\alpha_1, \alpha_2, \beta_1$ and β_2 are orthogonal idempotents.
Then $\alpha \otimes \beta = \alpha^2 \otimes \beta^2 = (\alpha \otimes \beta)(\alpha \otimes \beta) = (\alpha \otimes \beta)^2$. Hence, $\alpha \otimes \beta$ is an idempotent.
Also, $\alpha \otimes \beta \neq (\alpha_1 + \alpha_2) \otimes (\beta_1 + \beta_2) = (\alpha_1 + \alpha_2) \otimes \beta_1 + (\alpha_1 + \alpha_2) \otimes \beta_2$,
then $\alpha \otimes \beta \neq (\alpha_1 \otimes \beta_1) + (\alpha_2 \otimes \beta_1) + (\alpha_1 \otimes \beta_2) + (\alpha_2 \otimes \beta_2)$. Thus, $\alpha \otimes \beta$ is a primitive.
So, $P \otimes Q$ is a projective indecomposable $A_1 \otimes A_2$-module. □

The following theorem shows that, the tensor product of projective indecomposable modules is compatible with the Brauer linkage principle.

THEOREM 3.2.5. *([17], p.898). Let $P \otimes Q$, $P' \otimes Q'$ be two projective indecomposable $A_1 \otimes A_2$-modules. Then*

$$P \otimes Q \underset{A_1 \otimes A_2}{\approx} P' \otimes Q' \text{ if and only if } P \underset{A_1}{\approx} P' \text{ and } Q \underset{A_2}{\approx} Q'.$$

Proof. From theorem (1.2.4), let $P \otimes Q$, $P' \otimes Q'$ be two projective indecomposable $A_1 \otimes A_2$-modules.

$P \otimes Q \underset{A_1 \otimes A_2}{\approx} P' \otimes Q'$ if and only if there is a sequence from projective indecomposable $A_1 \otimes A_2$-modules $P \otimes Q = P_1 \otimes Q_1, P_2 \otimes Q_2, \ldots, P_t \otimes Q_t = P' \otimes Q'$; such that

$$(P_i \otimes Q_i, P_{i+1} \otimes Q_{i+1})_{A_1 \otimes A_2} \neq 0 \text{ or } (P_{i+1} \otimes Q_{i+1}, P_i \otimes Q_i)_{A_1 \otimes A_2} \neq 0.$$

From lemma (3.2.2), if and only if $(P_i, P_{i+1})_{A_1} \otimes (Q_i, Q_{i+1})_{A_2} \neq 0$ or $(P_{i+1}, P_i)_{A_1} \otimes (Q_{i+1}, Q_i)_{A_2} \neq 0$ for all $i = \{1, 2, \ldots, t\}$;
if and only if $(P_i, P_{i+1})_{A_1} \neq 0$ or $(P_{i+1}, P_i)_{A_1} \neq 0$ and $(Q_i, Q_{i+1})_{A_2} \neq 0$ or $(Q_{i+1}, Q_i)_{A_2} \neq 0$

for all $i = \{1, 2, \ldots, t\}$; where there is a sequence from projective indecomposable A_1-modules $P = P_1, P_2. \ldots, P_t = P'$ and there is a sequence from projective indecomposable A_2-modules $Q = Q_1, Q_2. \ldots, Q_t = Q'$;
if and only if $P \underset{A_1}{\approx} P'$ and $Q \underset{A_2}{\approx} Q'$. □

It follows that the tensor product of simple modules is compatible with the Brauer linkage principle as follows:

COROLLARY 3.2.6. *Let $X_1 \otimes X_2$, $Y_1 \otimes Y_2$ be two simple $A_1 \otimes A_2$-modules. Then*

$$X_1 \otimes X_2 \underset{A_1 \otimes A_2}{\approx} Y_1 \otimes Y_2 \text{ if and only if } X_1 \underset{A_1}{\approx} Y_1 \text{ and } X_2 \underset{A_2}{\approx} Y_2.$$

The following theorem shows that, the Brauer-Fitting correspondence is compatible with the tensor product of modules and algebras.

THEOREM 3.2.7. *([17], p.896). Let A_1, A_2 be two finite dimensonal algebras, and let Y_i be an A_i-module, where $i = 1, 2$.*
If $X_i \in Inds(Y_i)$ having Brauer-Fitting correspondents $S_i \in Irr(E(Y_i))$, then $X_1 \otimes X_2 \in Inds(Y_1 \otimes Y_2)$ with Brauer-Fitting correspondent $S_1 \otimes S_2 \in Irr(E(Y_1 \otimes Y_2))$.

Proof. The proof can be found in ([17], theorem1). □

The tensor operation is compatible with the pseudoblocks of the endomorphism algebra according to the following theorem.

THEOREM 3.2.8. *([17], p.896). Let A_1, A_2 be two finite dimensional algebras, Let Y_i be A_i-module, where $i = 1, 2$, and let $X_1 \otimes X_2$, $X_1' \otimes X_2' \in Inds(Y_1 \otimes Y_2)$. Then*

$$X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2' \text{ if and only if } X_1 \underset{PS(A_1)}{\approx} X_1' \quad \wedge \quad X_2 \underset{PS(A_2)}{\approx} X_2'.$$

Proof. Let $X_i, X_i' \in Inds(Y_i)$, where $\forall i = 1, 2$. Then X_i, X_i' have Brauer-Fitting Correspondents $S_i, S_i' \in Irr(E(Y_i))$, where$\forall i = 1, 2$.
Hence $(X_i, Y_i)_{A_i}, (X_i', Y_i)_{A_i}$, where $i = 1, 2$ are two projective indecomposable $E(Y_i)$-module.

Let $X_1 \underset{PS(A_1)}{\approx} X_1' \quad \wedge \quad X_2 \underset{PS(A_2)}{\approx} X_2'$
$\Leftrightarrow S_1 \underset{E(Y_1)}{\approx} S_1' \quad \wedge \quad S_2 \underset{E(Y_2)}{\approx} S_2'$
$\Leftrightarrow (X_1, Y_1)_{A_1} \underset{E(Y_1)}{\approx} (X_1', Y_1)_{A_1} \quad \wedge \quad (X_2, Y_2)_{A_2} \underset{E(Y_2)}{\approx} (X_2', Y_2)_{A_2}$ (by theorem (1.3.4))

$\Leftrightarrow (X_1, Y_1)_{A_1} \otimes (X_2, Y_2)_{A_2} \underset{E(Y_1 \otimes Y_2)}{\approx} (X_1', Y_1)_{A_1} \otimes (X_2', Y_2)_{A_2}$ (by lemma (3.2.4) and theorem (3.2.5)

$\Leftrightarrow (X_1 \otimes X_2, Y_1 \otimes Y_2)_{A_1 \otimes A_2} \underset{E(Y_1 \otimes Y_2)}{\approx} (X_1' \otimes X_2', Y_1 \otimes Y_2)_{A_1 \otimes A_2}$ (by lemma (3.2.2), (1))

$\Leftrightarrow S_1 \otimes S_2 \underset{E(Y_1 \otimes Y_2)}{\approx} S_1' \otimes S_2'$, (by theorem (3.2.7))

$\Leftrightarrow X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$, (by theorem (1.3.4)). □

The following theorem shows that the tensor operator on algebras and modules is compatible with the pseudoblock linkage principle.

THEOREM 3.2.9. *Let $A_1 \otimes A_2$ be a finite dimension algebra over the field F, where A_1, A_2 be two finite dimensional F-algebras; for each $X_1 \otimes X_2$, $X_1' \otimes X_2' \in Ind(A_1 \otimes A_2)$, where $X_1 \wedge X_1' \in IndA_1$ and $X_2 \wedge X_2' \in IndA_2$. Then*

$$X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2' \text{ if and only if } X_1 \underset{PS(A_1)}{\approx} X_1' \quad \wedge \quad X_2 \underset{PS(A_2)}{\approx} X_2'.$$

Proof. Let $A_1 \otimes A_2$ be a finite dimension algebra over the field F. For each $X_1 \otimes X_2$ and $X_1' \otimes X_2' \in Ind(A_1 \otimes A_2)$.

Suppose $X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$. Then from definition (1.1.1), there is a sequence of indecomposable modules

$$X_1 \otimes X_2 = U_1 \otimes V_1, U_2 \otimes V_2, \dots, U_t \otimes V_t = X_1' \otimes X_2',$$

in $Ind(A_1 \otimes A_2)$, then either

$$(U_j \otimes V_j, U_{j+1} \otimes V_{j+1})_{A_1 \otimes A_2} \neq 0 \quad \text{or} \quad (U_{j+1} \otimes V_{j+1}, U_j \otimes V_j)_{A_1 \otimes A_2} \neq 0,$$

for all $j = 1, 2, \dots, t$; from lemma (3.2.2),

$\Leftrightarrow (U_j, U_{j+1})_{A_1} \otimes (V_j, V_{j+1})_{A_2} \neq 0$ or $(U_{j+1}, U_j)_{A_1} \otimes (V_{j+1}, V_j)_{A_2} \neq 0$

$\Leftrightarrow$ $(U_j, U_{j+1})_{A_1} \neq 0$ and $(V_j, V_{j+1})_{A_2} \neq 0$ or $(U_{j+1}, U_j)_{A_1} \neq 0$ and $(V_{j+1}, V_j)_{A_2} \neq 0$ $\forall j = 1, 2, \dots, t$.

Where there is a sequence of indecomposable modules $X_1 = U_1, U_2, \dots, U_t = X_1'$ in $IndA_1$ and there is a sequence of modules $X_2 = V_1, V_2, \dots, V_t = X_2'$ in $IndA_2$ if and only if

$$X_1 \underset{PS(A_1)}{\approx} X_1' \quad \wedge \quad X_2 \underset{PS(A_2)}{\approx} X_2'.$$

□

Chapter 4

WORKED EXAMPLES

In this chapter, we investigate the pseudo-block distribution of the indecomposable modules for some known finite dimensional algebras. We shall give in detail the construction, and the composition factors of each indecomposable modules, and then determine their pseudo-block distributions. Because of the connection with the Brauer block theory, our pseudo-block distribution will be based on the (Brauer) block distribution.

4.1 SEMISIMPLE ALGEBRAS

In this section, we find that the regular left A-module ${}_AA$ is semisimple, and then A is semisimple as algebra. We also give the composition factors of each indecomposable modules, and then compare the two notions block and pseudoblock of semisimple algebras.

DEFINITION 4.1.1. *([10], p.91). Let A be an algebra over a field F. A is semisimple if the regular left A-module ${}_AA$ is semisimple.*

THEOREM 4.1.2. *([13], p.452). Let A be an F-algebra. Then every simple algebra A-module is a simple module over ring A.*

The following theorem shows that, the group algebra is semisimple if the characteristic of a field does not divide the order of a group.

THEOREM 4.1.3. *Let F be a field, let G be a group, and let FG be a group algebra. Then FG is semisimple if the characteristic of F does not divide the order of G according to Maschke's Theorem in ([19], p.21).*

THE ALGEBRA OF MATRICES $M_n(F)$.

Let $A = M_n(F)$ be an algebra of all $n \times n$ matrices over a field F. We show that A is semisimple algebra, and determine all finite dimensional A-modules (up to isomorphism).

We need to prove that A is semisimple algebra.

Step1. Take any finitely generated A-module V (notation: $V \in \text{mod}A$). Then V is finite dimension implies that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$, where $V_i \in IndA$.

Step2. Consider

$$S = F^n = \left\{ \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \alpha_i \in F \right\} = \{\alpha_1 e_1 + \ldots + \alpha_n e_n | \alpha_i \in F\};\ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

into, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$.

Define a left action of A on S by matrix multiplication. We note that S is simple A-module, for if not then $\exists 0 \neq W \leq S$, and so, $\exists 0 \neq \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in W; \alpha_i \neq 0_F \quad \forall i = \{1, 2, \ldots, n\}$.

Now, if we take:

$$a = \begin{array}{c} \text{ith position} \\ \downarrow \\ \begin{pmatrix} 0 & \alpha_i^{-1} & 0 \\ & & \\ 0 & & 0 \end{pmatrix}_{n\times n} \end{array} \in A, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \in S.$$

Then, $\begin{pmatrix} h_1 & 0 & \ldots & \ldots & 0 \\ h_2 & 0 & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ h_n & 0 & \ldots & \ldots & 0 \end{pmatrix} a\alpha = \begin{pmatrix} h_1 & 0 & \ldots & \ldots & 0 \\ h_2 & 0 & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ h_n & 0 & \ldots & \ldots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = h \in W.$

Therefore, $S \leq W$, and so, $S = W$.

Stepe3. We have ${}_AA \in modA \Rightarrow {}_AA = L_1 \oplus \ldots \oplus L_n$, where

$$L_\gamma = \left\{ \begin{pmatrix} 0 & \ldots & a_{1\gamma} & \ldots & 0 \\ 0 & \ldots & a_{2\gamma} & \ldots & 0 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & \ldots & a_{n\gamma} & \ldots & 0 \end{pmatrix}; a_{i\gamma} \in F \right\}.$$

Proof. Clear that $L_\gamma \leq {}_AA$, and for every $a = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \in A$;

we have $a = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21} & 0 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & a_{n2} & \ldots & 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 & 0 & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix} \in \sum_{i=1}^{n} L_i$; this representation is unique. Hence, ${}_AA = L_1 \oplus \ldots \oplus L_n$. □

Step4. Each $L_\gamma; \gamma = 1, 2, \ldots, n$, is simple A-module. In fact, $L_\gamma \cong S$. To prove that define $\lambda_\gamma : S \longrightarrow L_\gamma$ as follows:

$$\lambda_\gamma(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}) = \begin{pmatrix} 0 & \ldots & \alpha_{1\gamma} & \ldots & 0 \\ 0 & \ldots & \alpha_{2\gamma} & \ldots & 0 \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & \ldots & \alpha_{n\gamma} & \ldots & 0 \end{pmatrix}_{n\times n}$$

It is clear that $\lambda_\gamma : S \longrightarrow L_\gamma$ is a bijective map, because λ_γ is surjective; $\forall \begin{pmatrix} 0 & \dots & \alpha_{1\gamma} & \dots & 0 \\ 0 & \dots & \alpha_{2\gamma} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \alpha_{n\gamma} & \dots & 0 \end{pmatrix} \in L_\gamma, \quad \exists \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in S$ such that

$$\lambda_\gamma(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}) = \begin{pmatrix} 0 & \dots & \alpha_{1\gamma} & \dots & 0 \\ 0 & \dots & \alpha_{2\gamma} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \alpha_{n\gamma} & \dots & 0 \end{pmatrix}.$$

Also, λ_γ is injective; $\forall \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \in S$. Then,

$$\lambda_\gamma(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}) = \lambda_\gamma(\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix})$$

$$\begin{pmatrix} 0 & \dots & \alpha_{1\gamma} & \dots & 0 \\ 0 & \dots & \alpha_{2\gamma} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \alpha_{n\gamma} & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & \beta_{1\gamma} & \dots & 0 \\ 0 & \dots & \beta_{2\gamma} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \beta_{n\gamma} & \dots & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Thus, λ_γ is bijective. Also, λ_γ is an A-map, because

1.

$$\lambda_\gamma(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}+\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}) = \lambda_\gamma(\begin{pmatrix}\alpha_1+\beta_1\\ \alpha_2+\beta_2\\ \vdots\\ \alpha_n+\beta_n\end{pmatrix})$$

$$= \begin{pmatrix}0 & \dots & \alpha_{1\gamma}+\beta_{1\gamma} & \dots & 0\\ 0 & \dots & \alpha_{2\gamma}+\beta_{2\gamma} & \dots & 0\\ \vdots & \dots & \vdots & \dots & \vdots\\ 0 & \dots & \alpha_{n\gamma}+\beta_{n\gamma} & \dots & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & \dots & \alpha_{1\gamma} & \dots & 0\\ 0 & \dots & \alpha_{2\gamma} & \dots & 0\\ \vdots & \dots & \vdots & \dots & \vdots\\ 0 & \dots & \alpha_{n\gamma} & \dots & 0\end{pmatrix} + \begin{pmatrix}0 & \dots & \beta_{1\gamma} & \dots & 0\\ 0 & \dots & \beta_{2\gamma} & \dots & 0\\ \vdots & \dots & \vdots & \dots & \vdots\\ 0 & \dots & \beta_{n\gamma} & \dots & 0\end{pmatrix}$$

$$= \lambda_\gamma(\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}) + \lambda_\gamma(\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix})$$

2.

$$\lambda_\gamma(\mu\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}) = \lambda_\gamma(\begin{pmatrix}\mu\beta_1\\ \mu\beta_2\\ \vdots\\ \mu\beta_n\end{pmatrix}) = \begin{pmatrix}0 & \dots & \mu\beta_{1\gamma} & \dots & 0\\ 0 & \dots & \mu\beta_{2\gamma} & \dots & 0\\ \vdots & \dots & \vdots & \dots & \vdots\\ 0 & \dots & \mu\beta_{n\gamma} & \dots & 0\end{pmatrix}$$

$$= \mu\begin{pmatrix}0 & \dots & \beta_{1\gamma} & \dots & 0\\ 0 & \dots & \beta_{2\gamma} & \dots & 0\\ \vdots & \dots & \vdots & \dots & \vdots\\ 0 & \dots & \beta_{n\gamma} & \dots & 0\end{pmatrix} = \mu\lambda_\gamma(\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}),$$

where $\mu \in F$.

Therefore, $L_\gamma \cong S$ is simple A-module.
It follows that, ${}_AA = L_1 \oplus \ldots \oplus L_n$ is a direct sum of simple modules.
Therefore, ${}_AA$ is semisimple (as module). From definition (4.1.1), then A is semisimple as algebra. Hence, any $V \in modA$ is semisimple. From step1, then any indecomposable A-module is simple.

Step5. To find the simple A-module. Suppose that N is simple A-module. Choose $0 \neq n \in N$, and define $f_n : {}_AA \longrightarrow N$ by $(f_n(a) = an)$. It is clear that f_n is an A-epimorphism, because

1. f_n is an A-homomorphism, $\forall a, b \in {}_AA, \exists \alpha \in F$, where
   - $f_n(a+b) = n(a+b) = na + nb = f_n(a) + f_n(b)$;
   - $f_n(\alpha a) = n(\alpha a) = \alpha(na) = \alpha f_n(a)$.
2. f_n is surjective, i.e. $\forall y \in N, \exists a \in {}_AA,$ and $n \in N$, then $f_n(a) = na = y \in N$.

So, if $K = kerf_n$, then $K \leq A$. From First Isomorphism Theorem (0.1.19), $A/K \cong N$ is simple A-module. It follows that every simple A-module is isomorphic to a simple quotient (a composition factor) of ${}_AA$. But, we have the following **composition series** for ${}_AA$

$$ {}_AA = L_1 \oplus \ldots \oplus L_n \supset L_1 \oplus \ldots \oplus L_{n-1} \supset \ldots \supset L_1 \oplus L_2 \supset L_1. \tag{4.1} $$

And for each γ, we have

$$ L_1 \oplus \ldots \oplus L_\gamma / L_1 \oplus \ldots \oplus L_{\gamma-1} \cong L_\gamma \cong S. $$

And so, every **composition factor** of ${}_AA$ in (4.1) is simple. By Jordan-Holder Theorem (0.4.5), $A/K \cong S$, and $A/K \cong N$, hence $N \cong S$. So

1. **Every simple A-module is isomorphic to S.**
2. **Every A-module is isomorphic to direct sum of copies of S.**

Since, every A-module is isomorphic to direct sum of copies of S, and A is semisimple as algebra. Then the blocks of A are simple, hence all blocks do not split into union of pseudoblocks. Therfore, we have the following:

THEOREM 4.1.4. *For semisimple algebras the two notions (Blocks and Pseudoblocks) coincide.*

4.2 THE TRIANGULAR ALGEBRA A

$$ A = \{(a_{ij}) \in M_n(F) | a_{ij} = 0; \forall i > j\} = \left\{ a = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} ; a_{ij} \in F \right\} $$

is known to be of finite representation type (in fact uniserial algebra). We determine indecomposable module, projective indecomposable module (PIM), and the pseudoblock of A.

1. The algebra A acts on the space of column vectors

   $$ U = F^n = \left\{ \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} : t_i \in F; \forall i = 1, 2, \ldots, n \right\}; $$

   by usually matrix vector product.

   LEMMA 4.2.1. $N = \left\{ \begin{pmatrix} 0 & a_{12} & \dots & \dots & a_{1n} \\ & 0 & a_{23} & \dots & a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_{n-1n} \\ & & & & 0 \end{pmatrix} : a_{ij} \in F, i < j \right\} = rad(A),$
   where $radA$ is the radical of A, N is called strictly upper triangular matrix

   Proof. By definition(0.4.6), $rad(A)$ is the intersection of all the maximal submodule of A. Hence, the maximal submodules of A is strictly upper triangular matrix. Then, the radical of A is strictly upper triangular matrix. □

2.

LEMMA 4.2.2. *The triangular algebra has n simple (in fact 1-dimensional) representations.*

Proof. From theorem (0.4.10), $A/radA \cong socA = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$.

Hence,

$$socA = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$
$$= S_1 \oplus S_2 \oplus \dots \oplus S_n.$$

, where $S_i \quad \forall i = \{1, 2, \dots, n\}$; are simple A-modules. Then, A has n simple, where $\dim S_i = 1 \ \forall = 1, 2, \dots, n$. □

LEMMA 4.2.3. *Let F be a feild, and let A be a triangular matrix. Then $\psi_v : A \longrightarrow F \quad (a \longmapsto a_{vv})$, where $v = 1, 2, \dots, n$, is algebra map.*

Proof. For all $a, b \in A$ such that

$$a = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}; \forall a_{ij} \in F, \text{ and } b = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix}; \forall b_{ij} \in F,$$

then,

- $\psi_v(a + b) = (a_{vv} + b_{vv}) = \psi_v(a) + \psi_v(b)$;
- $\psi_v(ab) = (ab)_{vv} = a_{vv}b_{vv} = \psi_v(a)\psi_v(b)$;
- $\psi_v(\lambda a) = \lambda a_{vv} = \lambda\psi_v(a)$, where $\lambda \in F$;
- $\psi_v(1_A) = (1_A)_{vv} = 1 \in F$.

□

These representations are mutually inequivalent (if $\psi_v = \psi_\mu$, we should have Kerψ_v=Kerψ_μ, but clearly Kerψ_v=Ker$\psi_\mu \Rightarrow v = \mu$).

3. We have $NU = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ 0 \end{pmatrix} : v_i \in F \right\}$, and $N^iU = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_{n-i} \\ 0 \\ 0 \end{pmatrix} : v_i \in F \right\}$.

So $U \supset NU \supset \dots \supset N^{n-1}U \supset 0$ is a composition series with $dimN^{i-1}U/N^iU = 1; \forall i = 1, 2, \dots, n$. Also, $N^{i-1}U/N^iU$ affords the 1-dimension representation ψ_{n-i+1} of A; so the series

$$\begin{array}{c} U \supset NU \supset N^2U \supset \dots \supset N^{n-1}U \supset 0 \\ \psi_n \quad \psi_{n-1} \quad \psi_{n-2} \cdots \psi_2 \quad \psi_1 \end{array}$$

has composition factors as shown.

4. Since the series above is also the radical series of U, it is the only composition series for U. For if $U' \neq U$ is maximal in U, then $U' \geq NU$, but NU is also maximal in U. So $U' = NU$. Similarly if $U'' \neq NU$ is maximal in NU, we find that $U'' = N^2U$, $\ldots$, etc.

5. By (4), each A-module $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ has a unique composition series with composition factors of type $\psi_i, \psi_{i-1}, \psi_{i-2}, \ldots, \psi_{i-\alpha+1}$.

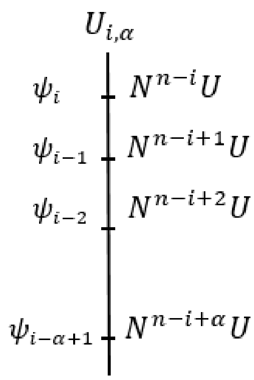

Figure 4.1:

It follows that $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ is indecomposable ($i = 1, 2, \ldots, n$ & $\alpha = 1, 2, \ldots, i$). (The number of such indecomposable module is $1 + 2 + 3 + \ldots + n = n(n+1)/2$ giving **a complete set of indecomposable A-modules**). Note that (by looking at the composition series), $U_{i,\alpha} \cong U_{j,\beta}$ iff $i = j$ and $\alpha = \beta$, and $dim U_{i,\alpha} = \alpha$. Also, $U_{1,1}$ affords the simple representation ψ_1, because $U_{1,1} = N^{n-1}U/N^nU = \psi_1$, in which $U_{i,1} = N^{n-1}U/N^{n-i+1}U = \psi_{n-(n-i)} = \psi_i$.

Summarizing we have the following:

THEOREM 4.2.4. *The A-module $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$; ($i = 1, 2, \ldots, n$ & $\alpha = 1, 2, \ldots, i$) give a complete set of $\dfrac{n(n+1)}{2}$ indecomposable A-module.*

6. Now, we can find the PIM's for A. We have $A = L_1 \oplus \ldots \oplus L_n$, where $L_v = \left\{ \begin{pmatrix} 0 & 0 & a_{1v} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & a_{vv} & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; a_{iv} \in F \right\}$. Clearly $L_v < A$ and $L_v \cong N^{n-v}U$ from (5), then $U_{v,v} = N^{n-v}U/N^nU \cong L_v$, i.e. (see (3)$\cong U_{v,v}$). Hence, $U_{1,1}, U_{2,2}, \ldots, U_{n,n}$ are a full set of PIM's. Note that, composition factors of $U_{v,v} = N^{n-v}U/N^nU = N^{n-v}U$ is as shown

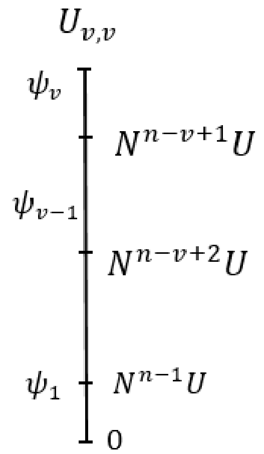

Figure 4.2:

Thus, we have the following result:

THEOREM 4.2.5. *The modules $U_{1,1}, U_{2,2}, \ldots, U_{n,n}$ give a full set of projective indecomposable A-modules.*

7. We already had $A = L_1 \oplus \ldots \oplus L_n = \sum_{1 \le v \le n} U_{v,v}$ (sum of PIM).
Since $rad(A) = N$ is non-trivial, A is not semisimple, because $rad(A) \neq 0$ from corollary (0.4.8), and so A has a non-trivial block theory. However, this algebra is known to be connected algebra; i.e. it has exactly one non-zero central idempotent, namely the identity matrix $n \times n$ (I_n). Hence,

THEOREM 4.2.6. *The triangular algebra A has a single block.*

EXAMPLE 4.2.7. *Take $n = 2$ it is easy to deduce the following:*

PROPOSITION 4.2.8. *Idempotents in $A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset M_2(F)$ are $\left\{ I_2, \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 0 & 1 \end{pmatrix} ; \lambda \in F \right\}$, where $*$ denotes elements in a field F. The only central idempotent of $A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is I_2.*

Then, A has $2(2+1)/2 = 3$ indecomposable modules $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ ($i = 1, 2$ & $\alpha = 1, 2, \ldots, i$), namely $U_{1,1} = \psi_1$, $U_{2,1} = \psi_2$, and $U_{2,2} = \begin{matrix} \psi_2 \\ \psi_1 \end{matrix}$ all lie in one pseudoblock, because there are A-module homomorphisms between all indecomposable modules as follows:

$$\begin{array}{ccc} & & U_{2,1} \\ & & \uparrow \\ U_{1,1} & \to & U_{2,2} \end{array}.$$

EXAMPLE 4.2.9. *Take $n = 3$, $A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \subset M_3(F)$. This algebra has $3(3+1)/2 = 6$ indecomposable modules $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ ($i = 1, 2, 3$ & $\alpha =$*

$1,2,\dots,i$), namely $U_{1,1}=\psi_1$, $U_{2,1}=\psi_2$, $U_{2,2}=\begin{matrix}\psi_2\\ \psi_1\end{matrix}$, $U_{3,1}=\psi_3$, $U_{3,2}=\begin{matrix}\psi_3\\ \psi_2\end{matrix}$, and $U_{3,3}=\begin{matrix}\psi_3\\ \psi_2\\ \psi_1\end{matrix}$ all lie in one pseudoblock, because there are A-module homomorphisms between all indecomposable modules as follows:

$$\begin{array}{ccccc} & & & & U_{3,1}\\ & & & & \uparrow\\ & & U_{2,1} & \to & U_{3,2}\\ & & \uparrow & & \uparrow\\ U_{1,1} & \to & U_{2,2} & \to & U_{3,3}\end{array}$$

EXAMPLE 4.2.10. *Take $n=4$, $A=\begin{pmatrix} * & * & * & *\\ 0 & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & 0 & *\end{pmatrix}\subset M_4(F)$. This algebra has $4(4+1)/2=10$ indecomposable modules $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$ ($i=1,2,3,4$ & $\alpha=1,2,\dots,i$), namely $U_{1,1}=\psi_1$, $U_{2,1}=\psi_2$, $U_{2,2}=\begin{matrix}\psi_2\\ \psi_1\end{matrix}$, $U_{3,1}=\psi_3$, $U_{3,2}=\begin{matrix}\psi_3\\ \psi_2\end{matrix}$, $U_{3,3}=\begin{matrix}\psi_3\\ \psi_2\\ \psi_1\end{matrix}$, $U_{4,1}=\psi_4$, $U_{4,2}=\begin{matrix}\psi_4\\ \psi_3\end{matrix}$, $U_{4,3}=\begin{matrix}\psi_4\\ \psi_3\\ \psi_2\end{matrix}$, $U_{4,4}=\begin{matrix}\psi_4\\ \psi_3\\ \psi_2\\ \psi_1\end{matrix}$, all lie in one pseudoblock, because there are A-module homomorphisms between all indecomposable modules as follows:*

$$\begin{array}{ccccccc} & & & & & & U_{4,1}\\ & & & & & & \uparrow\\ & & & & U_{3,1} & \to & U_{4,2}\\ & & & & \uparrow & & \uparrow\\ & & U_{2,1} & \to & U_{3,2} & \to & U_{4,3}\\ & & \uparrow & & \uparrow & & \uparrow\\ U_{1,1} & \to & U_{2,2} & \to & U_{3,3} & \to & U_{4,4}\end{array}$$

Now, we show in general that all the $n(n+1)/2$ indecomposable A-modules $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$; ($i=1,2,\dots,n$ & $\alpha=1,2,\dots,i$) are related (either ways) by homomorphisms:

Take any $n\in\mathbb{Z}$, $A=\begin{pmatrix} * & * & \dots & *\\ 0 & * & \dots & *\\ 0 & 0 & \ddots & \vdots\\ 0 & 0 & 0 & *\end{pmatrix}\subset M_n(F)$. This algebra has $n(n+1)/2$ indecomposable modules $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$ ($i=1,2,\dots,n$ & $\alpha=1,2,\dots,i$), namely $U_{1,1}=\psi_1$, $U_{2,1}=\psi_2$, $U_{2,2}=\begin{matrix}\psi_2\\ \psi_1\end{matrix}$, $U_{3,1}=\psi_3$, $U_{3,2}=\begin{matrix}\psi_3\\ \psi_2\end{matrix}$, $U_{3,3}=\begin{matrix}\psi_3\\ \psi_2\\ \psi_1\end{matrix}$, $U_{4,1}=\psi_4$, $U_{4,2}=\begin{matrix}\psi_4\\ \psi_3\end{matrix}$, $U_{4,3}=\begin{matrix}\psi_4\\ \psi_3\\ \psi_2\end{matrix}$, $U_{4,4}=\begin{matrix}\psi_4\\ \psi_3\\ \psi_2\\ \psi_1\end{matrix}$, into $U_{n,1}=\psi_n$, $U_{n,2}=\begin{matrix}\psi_n\\ \psi_{n-1}\end{matrix}$, $U_{n,3}=\begin{matrix}\psi_n\\ \psi_{n-1}\\ \psi_{n-2}\end{matrix}$, into

$U_{n,\alpha} = \begin{matrix} \psi_n \\ \psi_{n-1} \\ \vdots \\ \psi_{n-\alpha+1} \end{matrix}$, into $U_{n,n} = \begin{matrix} \psi_n \\ \psi_{n-1} \\ \vdots \\ \psi_1 \end{matrix}$ all lie in one pseudoblock, because there are A-module homomorphisms between all indecomposable modules as follows:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & U_{n,1} \\
 & & & & & & & & & & \uparrow \\
 & & & & & & & & & & \vdots \\
 & & & & & & & & & & \uparrow \\
 & & & & & & U_{4,1} & \rightarrow & \dots & \rightarrow & U_{n,n-3} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & U_{3,1} & \rightarrow & U_{4,2} & \rightarrow & \dots & \rightarrow & U_{n,n-2} \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & U_{2,1} & \rightarrow & U_{3,2} & \rightarrow & U_{4,3} & \rightarrow & \dots & \rightarrow & U_{n,n-1} \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
U_{1,1} & \rightarrow & U_{2,2} & \rightarrow & U_{3,3} & \rightarrow & U_{4,4} & \rightarrow & \dots & \rightarrow & U_{n,n}
\end{array}$$

8. Hence, all these modules lie in the same pseudoblock. So, we have the following:

THEOREM 4.2.11. *For the triangular algebra, the block and pseudoblock notions coincide.*

4.3 FS_3 IN ALL CHARACTERISTICS

In this section, we explain the pseudoblock structure for the symmetric group algebra $\Lambda = FS_3$ over a field F in all characteristics, and then compare the two notions "block" and "pseudoblock" in the category of FS_3-modules.

First: Representation of $G = S_3$ in characteristic zero and characteristic prime numbers ($p' \geq 5$); from Maschke's theorem in ([19], p.21), then FS_3 completely reducible (semisimple); because $0 \nmid 6$, $5 \nmid 6$, $7 \nmid 6$, $\dots$ and $p' \nmid |S_3|$.
From section (4.1), then for FS_3 the two notions (blocks and pseudoblocks) coincide.

Second: Representation of $G = S_3$ in characteristic 2. It is known that $S_3 \cong SL(2,2) \cong GL(2,2) =< a, b | a^3 = b^2 = 1, bab^{-1} = a^{-1} >$.
Therefor, **in characteristic** 2; from theorem (0.1.23), S_3 has two classes of simple module (it has 2 2-regular conjugacy classes $< 1 >, < a >$ from example (0.1.24)); namely the trivial module F_G and the 2-dimensional simple module St_G (The Steinberg representation). It is known that S_3 in characteristic 0 has 3 irreducible characters (χ_1: the trivial, χ_2: the sing, and χ_3: the Steinberg characters). The sign and the trivial character coincide over the field of characteristic 2 and Steinberg character remains irreducible. So, the character table of S_3 as stated in ([19], p.50),

	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

This group has two 2-modular (Brauer) characters ϕ_1, ϕ_2, where ϕ_1 is the trivial character, and ϕ_2 is the Steinberg character,

	1	a
ϕ_1	1	1
ϕ_2	1	1

Then, the decomposition matrix of S_3 is $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and hence the Cartan matrix is $C = DD^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.
Hence, it has 2 blocks; from theorem (2.1.1), hence at least two pseudoblocks.
Then

$$FS_3 = \frac{F_G}{F_G} \oplus St_G \oplus St_G.$$

REMARK. *FS_3 is finite representation type, because Sylow 2-subgroup of S_3 $\{< b >, < a^2b >, < ab >\}$ is cyclic of order two. Hence, the number of indecomposable module is* $(p-1)^2 + 2$.

Then, this group algebra has $(2-1)^2 + 2 = 3$ classes of indecomposable module two of them are projective $\frac{F_G}{F_G}$, St_G and the trivial module F_G is the third indecomposable module.
Now, we find clearly that the pseudoblock distribution of the indecomposable modules as following: $\{F_G, \frac{F_G}{F_G}\}$, and $\{St_G\}$ which is identical with the (Brauer) block distribution.

Third: Representation of $G = S_3$ in characteristic 3.
Let $G = S_3$ and take F to be a field of characteristic 3. It is known, by Higman's criterion (0.3.7) that the group algebra FG is of finite representation type (Since its Sylow 3-subgroup is cyclic from example (0.3.5)). We shall **construct the complete set of indecomposable FG-modules for S_3**. This comes out as a special case of a general algorithm by theorem (0.3.10) for constructing **the complete set of indecomposable FG-module for a finite group G with cyclic normal Sylow p-subgroup.** We have $G = S_3 =< a, b | a^3 = b^2 = 1, bab^{-1} = a^{-1} >$ and $H =< a > \in Syl_3(G)$.
Also, H is normal subgroup of G. The group algebra

$$FH =< a-1 | (a-1)^3 = 0 >= F.1 \oplus F.(a-1) \oplus F.(a-1)^2,$$

$(\forall x \in FG, x = \lambda_0 1 + \lambda_1 a + \lambda_2 a^2 = \lambda_0 1 + (\lambda_1 - \lambda_2)(a-1) + (\lambda_0 + \lambda_1 - \lambda_2)(a-1)^2)$ has the following unique composition series (since FH is projective) as a left FH-module

$$\begin{array}{ccccccc} FH & \supset & (a-1)FH & \supset & (a-1)^2FH & \supset & 0 \\ & & \| & & \| & & \\ & & F.(a-1) \oplus F.(a-1)^2 & & F(a-1)^2 & & \end{array} \tag{4.2}$$

Putting $W_r = FH/(a-1)^r FH$, $r = 1, 2, 3$, we see (from a general theory (0.3.10)) that $W_r \in IndFH$, and that $\{W_i | i = 1, 2, 3\}$ is a full set of indecomposable FH-module with $\dim_F W_r = r$.

Note: Since H is 3-group and F is of characteristic 3; it follows that H has only one simple module namely the trivial FH-module F_H from corollary (0.3.8), and $W_3 \cong FH$ is the (only) projective indecomposable FH-module, which is the projective cover of F_H.

Define $e_0 = \frac{1}{2}(1+b), e_1 = \frac{1}{2}(1-b) \in FG$. Then $e_i^2 = e_i; i = 0,1$ and

$$be_0 = e_0 \quad \& \quad be_1 = -e_1 \qquad [i.e. \quad be_j = (-1)^j e_j] \tag{4.3}$$

$$\& \quad b(a-1)^r = b(a-1)^r b^{-1} b = (a^{-1}-1)^r b = (a^2-1)^r b = (a+1)^r (a-1)^r b.$$

Also, $1 = e_0 + e_1$ and $e_0 e_1 = 0$. Therefore, $FG = FGe_0 \oplus FGe_1$. Write $V_j = FGe_j$; $j = 0,1$. Then, from the actions in (4.3), we see that $V_j = FGe_j \cong FHe_j$ has an F-basis $\{e_j, (a-1)e_j, (a-1)^2 e_j\}$. Clearly, ${}_{FH}FH \to V_j, (\alpha \longmapsto \alpha e_j; \alpha \in FH)$ defines an FH-isomorphism, and so $(V_j)_H \cong FH$, which means (since $H \in Syl_3(G)$) that $V_j \in$IndFG. Note that, we can easily see that $V_0 = \mathrm{ind}_{<b>}^G k = FG \otimes_{F<b>} k$ and $V_1 = \mathrm{ind}_{<b>}^G \varepsilon = FG \otimes_{F<b>} \varepsilon$. Next, from (4.3), $(a-1)^r V_j = (a-1)^r FHe_j$ is an FG-module, and hence

$$V_j \supset (a-1)V_j \supset (a-1)^2 V_j \supset 0$$

is a (unique) FG-composition series for V_j.

We are now ready to give the full set of indecomposable FG-module. Write $V_{j,r} = V_j/(a-1)^r V_j$; $j = 0,1$ & $r = 1,2,3$

$$V_j \supset (a-1)V_j \supset (a-1)^2 V_j \supset 0. \tag{4.4}$$

<u>Note:</u> G has two (3-regular conjugacy classes; namely $< 1 >, < b >$) simple modules each of dimension one, let S_0 and S_1.
The character table of S_3 as stated in ([19], p.50) is

	1	a	b
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

This group has two 3-modular (Brauer) characters ϕ_1, ϕ_2, where ϕ_1 is the trivial character and ϕ_2 is the Steinberg character given by the table

	1	b
ϕ_1	1	1
ϕ_2	1	-1

Then, the decomposition matrix of FS_3 is $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, and hence its Cartan matrix is $C = DD^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which indicate the dimensions and composition factors of each projective indecomposable module (i.e. it has one block).

Summarizing, we have the following theorem:

THEOREM 4.3.1. *1.* $\{V_{j,r} \mid j = 0,1$ *&* $r = 1,2,3\ \}$ *is a full set of* $IndFG$.

2. $V_{j,r}$ *is projective* $\Leftrightarrow r = 3$, *in which case* $V_{j,3} \cong V_j$.

3. $\{S_j = V_{j,1} \mid j = 0,1\}$ *is full set of simple FG-modules.*

4. Composition factors of $V_0, V_1, V_{1,2}$, and $V_{0,2}$ are as shown

$$V_0 \sim \begin{array}{c} S_0 \\ S_1, \\ S_0 \end{array} \quad V_1 \sim \begin{array}{c} S_1 \\ S_0, \\ S_1 \end{array} \quad V_{1,2} \sim \begin{array}{c} S_1 \\ S_0 \end{array}, \quad V_{0,2} \sim \begin{array}{c} S_0 \\ S_1 \end{array},$$

Hence indecomposable (6 isomorphism classes) are determined by sequence of composition factors concerned modules.

Proof. **1. $\{V_{j,r} \mid j = 0,1\ \&\ r = 1,2,3\ \}$ is a full set of $IndFG$.**

Let F be a field of characteristic 3, and let G be a finite group with normal Sylow 3-subgroup H. Then from ([6], proposition20.13 (ii), p.479) the number of indecomposable FG-module is $|G| = 6$. Thus, indecomposable FG-modules are $V_{0,1}, V_{0,2}, V_{0,3}, V_{1,1}, V_{1,2}$ and $V_{1,3}$. So $\{\ V_{j,r} \mid j = 0,1\ \&\ r = 1,2,3\ \}$ is a full set of $IndFG$, where the $\dim V_{j,r} = r$ for all j, r.

2. $V_{j,r}$ is projective $\Leftrightarrow r = 3$, in which case $V_{j,3} \cong V_j$.

Let $r = 3$ in which case $V_{j,3} \cong V_j$

$\Leftrightarrow FG \cong V_0 \oplus V_1$

$\Leftrightarrow FG \cong \dfrac{V_0}{(a-1)^3 V_0} \oplus \dfrac{V_1}{(a-1)^3 V_1}$, where $(a-1)^3 = 0$

$\Leftrightarrow FG \cong V_{0,3} \oplus V_{1,3}$

$\Leftrightarrow V_{j,3}$ is projective, because $V_{j,3}$ is direct summand of FG $(V_{j,3}|FG)$. Hence, $V_{j,r}$ is projective at $r = 3$, where $j = 0, 1$.

3. **$\{S_j = V_{j,1} \mid j = 0,1\}$ is full set of simple FG-modules.**
The conjugacy classes of S_3 are $\{1\}$, $\{b, ab, a^2b\}$ and $\{a, a^2\}$. Hence, there exist two 3-regular conjugacy classes of S_3. Then from theorem (0.1.23), there are two simple FG-modules S_0 and S_1, where $\dim S_j = 1$ from ([6], proposition20.13 (i), p.479). Then $V_{1,1} = \dfrac{V_1}{(a-1)V_1} = S_1$ and $V_{0,1} = \dfrac{V_0}{(a-1)V_0} = S_0$; i.e. $\{S_j = V_{j,1} | j = 0,1\}$ is full set of simple FG-module.

4. **Composition factors of $V_0, V_1, V_{1,2}$, and $V_{0,2}$ are as shown**

$$V_0 \sim \begin{array}{c} S_0 \\ S_1, \\ S_0 \end{array} \quad V_1 \sim \begin{array}{c} S_1 \\ S_0, \\ S_1 \end{array} \quad V_{1,2} \sim \begin{array}{c} S_1 \\ S_0 \end{array}, \quad V_{0,2} \sim \begin{array}{c} S_0 \\ S_1 \end{array},$$

Hence, indecomposable (6 isomorphism classes) are determined by sequence of composition factors concerned modules.

From 2., V_0 and V_1 are projective indecomposable FG-module. From proposition (0.3.9), then $\dim V_0 = 3$ and $\dim V_1 = 3$, because Sylow 3-subgroup H is order 3, then every projective FG-module has dimension divisible by 3, in which $|FG| = 6$, hence $\dim V_0 = 3$ and $\dim V_1 = 3$.
From (4.4), then the structure of complete set of indecomposable FG-modules are

$$V_0 = V_{0,3} = \frac{V_0}{(a-1)^3 V_0} \sim \begin{matrix} S_0 \\ S_1, \\ S_0 \end{matrix}$$

also, $V_1 = V_{1,3} = \dfrac{V_1}{(a-1)^3 V_1} \sim \begin{matrix} S_1 \\ S_0; \\ S_1 \end{matrix}$

from 3., then $V_{1,1} = \dfrac{V_1}{(a-1)V_1} = S_1$ and $V_{0,1} = \dfrac{V_0}{(a-1)V_0} = S_0$;

from 1., $V_{0,2} = \dfrac{V_0}{(a-1)^2 V_0} \sim \begin{matrix} S_0 \\ S_1 \end{matrix}$, where $\dim V_{0,2} = 2$,

and $V_{1,2} = \dfrac{V_1}{(a-1)^2 V_1} \sim \begin{matrix} S_1 \\ S_0 \end{matrix}$, where $\dim V_{1,2} = 2$.

$\square$

From the Cartan matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, FG has one block (the principle block B_0) with 2 projective indecomposable modules $V_0 = \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix}$ and $V_1 = \begin{matrix} S_1 \\ S_0, \\ S_1 \end{matrix}$ and we have the following decomposition ${}_{FS_3}FS_3 = V_0 \oplus V_1 = \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix} \oplus \begin{matrix} S_1 \\ S_0 \\ S_1 \end{matrix}$; i.e. every indecomposable FG-module in the same block. In fact, FG also has one pseudoblock, because there exist FG-module homomorphisms between all indecomposable FG-modules as follows:

$$S_0 \leftarrow \begin{matrix} S_0 \\ S_1 \end{matrix} \leftarrow \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix} \rightarrow \begin{matrix} S_1 \\ S_0 \\ S_1 \end{matrix} \rightarrow \begin{matrix} S_1 \\ S_0 \end{matrix} \rightarrow S_1.$$

Summarizing, we have the following conclusion:

THEOREM 4.3.2. *The blocks and pseudoblocks coincide for the group algebra FS_3 for any field F.*

4.4 THE CASE $\Lambda = FC_n$; $n = p^a e$; $p \nmid e$.

Here, we shall construct a complete set of indecomposable FC_n-modules, and then compare the two notions "blocks" and "pseudoblocks" in the category of FC_n-modules. Since FC_n is semisimple when the characteristic of F ($CharF$) $\nmid n$, it is enough to consider the case when $CharF = p$.

It is known (see [2], p.24, p.25, and p.34) that $\Lambda = FG$; $G = C_n$ has e simple modules, because G has n conjugacy classes, but G has e elements of order is not divisible by p; then, G has e p-regular conjugacy classes from theorem (0.1.23). Also, FG has a total of $n = p^a e$ indecomposable modules, because F is a field of characteristic p, and C_n is finite group with cyclic Sylow p-subgroup P, then from ([6], proposition20.13, p.479), the number of indecomposable FG-modules is $|C_n| = n = p^a e$.

We now describe the ($n = p^a e$) indecomposable (of which e are simple) FG-modules. There are e simple modules $\{S_\lambda | \lambda \text{ is an } e\text{-th root of } 1\}$ are all 1-dimensional, where $S_\lambda =$

F on which G acts by multiplication with λ. For each integer $1 \leq m \leq p^a$, there is a uniserial module $L_{\lambda,m}$ of dimension m with each composition factor isomorphic to S_λ (note that $L_{\lambda,1} = S_\lambda$).

Consequently, the group algebra FG has $(g^e - 1)$ is nilpotent element, hence $N = (g^e - 1)$ is radFG from definition (0.4.6) and from definition(0.4.12). So, we have the radical series

$$L_\lambda \supset NL_\lambda \supset N^2L_\lambda \supset ... \supset N^{p^a}L_\lambda = 0. \tag{4.5}$$

Thus, L_λ/N^mL_λ is composition factor $\forall\lambda = 1, ..., e$, and $\forall m = 1, 2, ..., p^a$. So, $L_{\lambda,m} = L_\lambda/N^mL_\lambda$, $\forall\lambda = 1, 2, ..., e$ and $\forall m = 1, 2, ..., p^a$ given all indecomposable FG-modules. Accordingly, The set $\{L_{\lambda,m}|\lambda, m\}$ gives a complete set of $n = p^ae$ indecomposable FG-modules, and $\{L_{\lambda,p^a}|\lambda\}$ are the complete set of projective indecomposable FG-modules (L_{λ,p^a} is the projective cover of S_λ). Then,
first: the structure of all simple FG-modules are $S_1 = L_{1,1}$, $\quad S_2 = L_{2,1}$, $\quad$ into, $S_e = L_{e,1} = \frac{L_e}{NL_e}$, where $\dim S_\lambda = 1$.

Second: the structure of all projective indecomposable FG-modules are

$$L_{1,p^a} \sim \begin{matrix} S_1 \\ S_1 \\ \vdots \\ S_1 \end{matrix}, \quad L_{2,p^a} \sim \begin{matrix} S_2 \\ S_2 \\ \vdots \\ S_2 \end{matrix}, \quad \text{into } L_{e,p^a} = \frac{L_e}{N^{p^a}L_e} \sim \begin{matrix} S_e \\ S_e \\ \vdots \\ S_e \end{matrix}, \text{ where } \dim L_{\lambda,p^a} = p^a.$$

Finally: the structure of all non-simple and non-projective indecomposable FG-modules are

$$L_{\lambda,2} \sim \begin{matrix} S_\lambda \\ S_\lambda \end{matrix}, \qquad L_{\lambda,3} \sim \begin{matrix} S_\lambda \\ S_\lambda \\ S_\lambda \end{matrix}, \qquad \text{into } L_{\lambda,p^a-1} = \frac{L_\lambda}{N^{p^a-1}L_\lambda} \sim \begin{matrix} S_\lambda \\ S_\lambda \\ \vdots \\ S_\lambda \end{matrix}, \text{ where } \dim L_{\lambda,p^a-1} = p^a - 1.$$

We also have $FG = FC_n = \sum_\lambda^\oplus L_{\lambda,p^a}$. Clearly, $FG = FC_n$ has e Brauer p-blocks $\{B_\lambda\}_{1\leq\lambda\leq e}$, where $B_\lambda = \Big\{L_{\lambda,m}; 1 \leq m \leq p^a\Big\}_{1\leq\lambda\leq e}$. All indecomposable FC_n-modules in the block B_λ are uniserial with all of its composition factors isomorphic to $L_{\lambda,1} = S_\lambda$. Therefore, they all lie in the one pseudoblock B_λ, because there exist FG-module homomorphisms between all indecomposable FG-modules in B_λ, $\forall\lambda = \{1, 2, 3, \dots, e\}$ as follows:

$$S_\lambda \to \begin{matrix} S_\lambda \\ S_\lambda \end{matrix} \to \begin{matrix} S_\lambda \\ S_\lambda \\ S_\lambda \end{matrix} \to \dots \to \begin{matrix} S_\lambda \\ S_\lambda \\ S_\lambda \\ \vdots \\ S_\lambda \end{matrix}.$$

EXAMPLE 4.4.1. *Take $\Lambda = FC_6; 6 = 3.2$, $CharF = 3$. We have 2 simple modules S_1, S_{-1} among 6 uniserial indecomposable FC_6-modules $L_{1,1} = S_1, L_{1,2} \sim \begin{matrix} S_1 \\ S_1 \end{matrix}, L_{1,3} \sim \begin{matrix} S_1 \\ S_1 \\ S_1 \end{matrix}, L_{-1,1} = S_{-1}, L_{-1,2} \sim \begin{matrix} S_{-1} \\ S_{-1} \end{matrix}, L_{-1,3} \sim \begin{matrix} S_{-1} \\ S_{-1} \\ S_{-1} \end{matrix}$. It is clear that FC_6 has 2 pseudoblocks $L_{1,1} = S_1, L_{1,2}, L_{1,3}$ and $L_{-1,1} = S_{-1}, L_{-1,2}, L_{-1,3}$.*

Therefore, we have the following:

THEOREM 4.4.2. *If $\Lambda = FG, G = C_n$; $n = p^a e$; $p \nmid e$ in characteristic p, then $\Lambda = FG$ has e pseudo-blocks. Hence, the blocks and pseudoblocks, coincide in this case.*

4.5 p-GROUP ALGEBRA

Let F be a field, let G be a p-group, and let $\Lambda = FG$ be a p-group algebra.

If the characteristic of F is not equal to p (i.e. char$F \nmid |G|$), then FG is completely reducible (semisimple) according to Maschke's theorem in ([19], p.21).
Thus, from section (4.1), then for FG the two notions blocks and pseudoblocks coincide.

While if the characteristic of F is equal to p, and G is a p-group, then the only simple FG-module is the trivial module F_G, where dim$F_G = 1$, as stated in the corollary (0.3.8).
Hence, the group algebra FG contains one projective indecomposable module.
Then, the construction of all indecomposable FG-modules contains the trivial composition factor F_G. Clearly, the p-group algebra FG has a single block B, which contains all indecomposable FG-modules.
Hence, there exist FG-module homomorphisms between all indecomposable FG-modules in B as follows:

$$F_G \to \begin{matrix} F_G \\ F_G \end{matrix} \to \begin{matrix} F_G \\ F_G \\ F_G \end{matrix} \to \ldots \to \begin{matrix} F_G \\ F_G \\ F_G \\ \vdots \\ F_G \end{matrix}.$$

Then, the block B does not split into union of pseudoblocks. We summarize this result as follows:

THEOREM 4.5.1. *The blocks and pseudoblocks coincide for the p-group algebra FG for any field F.*

Combining Theorems (4.4.2), (4.5.1) above we have

THEOREM 4.5.2. *Let $\Lambda = FC_q; q = p^a e$. Then $Ind\Lambda / \underset{\Lambda}{\approx} = Ind\Lambda / \underset{PS\Lambda}{\approx}$ for any field F.*

Chapter 5

THE PSEUDOBLOCKS OF $FSL(2,p)$ IN CHARACTERISTIC p

In this chapter, we shall determine the pseudoblocks of the group algebra $FSL(2,p)$, where $Char F = p$. We shall follow the paper of D. Craven [4] in which he gave a full description for the complete set of indecomposable $FSL(2,p)$-modules. We shall compare the block and pseudoblock theory of the group algebra $FSL(2,p)$.

5.1 THE SIMPLE $FSL(2,p)$-MODULES

It is known that if p is odd prime, then $G = SL(2,p)$ has $p+4$ conjugacy classes, where G has p conjugacy classes which are p-regular while if $p = 2$, then G has $p+1$ conjugacy classes, where G has p conjugacy classes which are p-regular as stated in ([8], §38) and ([2], p.14); so the number of simple $FSL(2,p)$-modules is p. We now describe those simple modules; G acts on the space of column vectors V_2 over F ($= F^2$); hence V_2 is an FG-module. We write $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and so if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then $gX = aX + cY$ and $gY = bX + dY$. Hence, the action of each $g \in G = SL(2,p)$ extends to an automorphism of the polynomial ring $F[X,Y]$. Let V_n be the subspace of $F[X,Y]$ consisting of homogeneous polynomials in X, Y of degree $n-1$. In particular V_n is an $FSL(2,p)$-module (V_2 is as before). $\mathrm{Dim}_F V_n = n$, and has an F-basis consist of the polynomials $X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1}$.

We shall now prove that V_{n+1} is simple at $1 \leqslant n < p$. Let $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$.
We shall consider V_{n+1} is an $F < g >$-module and V_{n+1} is an $F < h >$-module. Let X^n be a generator of the $F < g >$-module V_{n+1}, and that the subspace FY^n is the socle, where FY^n spanned by Y^n.
Also, Y^n is a generator of the $F < h >$-module V_{n+1}, and FX^n is the socle as stated in ([2], p.15).
Suppose that W is a non-zero submodule of V_{n+1},

$$W \leqslant V_{n+1}. \tag{5.1}$$

Hence, W is an $F < g >$-submodule. So, W contains of a simple $F < g >$-submodule, but FY^n is only simple $F < g >$-submodule in V_{n+1}, so $Y^n \in W$.
Therefore, the $F < h >$-module V_{n+1} generated by Y_n (i.e. $Y^n \in V_{n+1}$) is also contained in W

$$V_{n+1} \leqslant W. \tag{5.2}$$

From (5.1) & (5.2), then $W = V_{n+1}$. Hence, for all $1 \leqslant n < p$, V_{n+1} are simple $FSL(2,p)$-modules.

Therefore, we have the following:

THEOREM 5.1.1. *([2], p.15). The $FSL(2,p)$-modules $V_1, V_2, \ldots, V_p$ are the complete set of simple FG-modules, where V_1 is the trivial $FSL(2,p)$-module, and V_p is projective indecomposable $FSL(2,p)$-module (the Steinberg representation). Also, $dim_F V_n = n$ for all $1 \leq n \leq p$.*

We can classify all simple $FSL(2,p)$-modules as follows:

LEMMA 5.1.2. *The odd-dimensional simple modules are modules for projective linear group $PSL(2,p)$, and the even-dimensional simple modules are faithful modules for $SL(2,p)$.*

Proof. The normal subgroup of $G = SL(2,p)$ is the center of $SL(2,p)$;

$$Z(SL(2,p)) = \{A \in SL(2,p) : AT = TA \quad \forall T \in SL(2,p)\}.$$

Then, $Z(SL(2,p)) = \{\pm I_2\} \lhd SL(2,p)$. Hence, the projective linear group $PSL(2,p) = SL(2,p)/Z(SL(2,p))$.

We use Lifting Process theorem (0.1.15), hence

let $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} aX + cY = \begin{pmatrix} aX + cY & 0 \\ 0 & aX + cY \end{pmatrix}$. Then

$$\begin{aligned} det \begin{pmatrix} aX + cY & 0 \\ 0 & aX + cY \end{pmatrix} &= (aX + cY)(aX + cY) \\ &= a^2X^2 + 2acXY + c^2Y^2; \end{aligned}$$

so $\{X^2, XY, Y^2\}$ is the basis of V_3.

Also, $\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} V_2 \in V_3$, (i.e. V_3 contains $\det(ZV_2)$, where $Z = Z(SL(2,p))$).

Let

$$\begin{aligned} det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_3 &= (X^2 + XY + Y^2)(X^2 + XY + Y^2) \\ &= X^4 + 2X^3Y + 3X^2Y^2 + 2XY^3 + Y^4; \end{aligned}$$

so $\{X^4, X^3Y, X^2Y^2, XY^3, Y^4\}$ is the basis of V_5.
Also, $\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} V_3 \in V_5$, (i.e. V_5 contains $\det(ZV_3)$).

Let

$$\begin{aligned} det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_i &= (X^{i-1} + X^{i-2}Y + \ldots + Y^{i-1})(X^{i-1} + X^{i-2}Y + \ldots + Y^{i-1}) \\ &= X^{2i-2} + X^{2i-3}Y + \ldots + X^{i-1}Y^{i-1} + \ldots + X^{i-1}Y^{i-1} + X^{i-2}Y^{i} + \\ &\quad \ldots + Y^{2i-2}; \end{aligned}$$

so $\{X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1}\}$ is the basis of V_j.

Then, V_j contains $\det Z V_i$ for all $i = 1, 2, \ldots, p$, where j is odd number. Then $PSL(2,p)$ contains the odd-dimensional simple modules, but the even-dimensional simple modules are faithful modules for $SL(2,p)$. □

More details on $PSL(2,p)$ are given in ([8], §35).

5.2 THE PROJECTIVE INDECOMPOSABLE $FSL(2,p)$-MODULES

From theorem (5.1.1), the group algebra $FSL(2,p)$ has p simple FG-modules, and they are $V_1, V_2, \ldots, V_p$, where V_1 is the trivial FG-module and V_p is the Steinberg module, in which $\dim V_n = n$, for all $1 \leq n \leq p$. It follows that, $FSL(2,p)$ has p projective indecomposable modules, because all projective indecomposable FG-modules corresponding to simple FG-modules. The indecomposable $FSL(2,p)$-modules $P_1, P_2, \ldots, P_p$ are the complete set of projective indecomposable FG-modules, where $P_p \cong V_p$.

Now, we determine the projective indecomposable FG-modules from ([2], p.48, p.51 and p.78).

P_1 is uniserial from theorem (0.5.4), then $\dfrac{rad(P_1)}{soc(P_1)} \cong V_{p-2}$ if and only if dim $P_1 \geqslant 1 + (p-2) + 1 = p$, hence the structure of P_1 is

$$\begin{array}{c} V_1 \\ | \\ V_{p-2} \\ | \\ V_1 \end{array},$$

If $p = 2$, then P_1 has two factors V_1 and V_1.

P_{p-1} is also uniserial. If $p > 2$, then from theorem (0.5.5), hence $\dfrac{rad(P_{p-1})}{soc(P_{p-1})} \cong V_2$ if and only if dim $P_{p-1} = (p-1) + 2 + (p-1) = 2p$.

Then, the structure of P_{p-1} if $p > 2$ is

$$\begin{array}{c} V_{p-1} \\ | \\ V_2 \\ | \\ V_{p-1} \end{array}.$$

$V_2 \otimes V_p$ is projective by theorem (0.5.6) and from theorem (0.5.7); $V_2 \otimes V_p$ has a submodule isomorphic with V_{p-1}, also the quotient of this submodule is isomorphic with V_{p+1}.

We know that, every projective submodule of a module M is a direct summand of M, then $V_2 \otimes V_p$ has direct summand which is indecomposable, and it has submodule isomorphic with V_{p-1}, where V_{p-1} is socle of P_{p-1}, then $V_2 \otimes V_p$ has a summand isomorphic with P_{p-1}. P_{p-1} has dimension at least $2(p-1)$, but P_{p-1} has dimension $2p$ at $p > 2$, and $V_2 \otimes V_p$ is $2p$-dimension, then

$$V_2 \otimes V_p \cong P_{p-1}.$$

Let $V = soc(V_2 \otimes V_p) \cong V_{p-1}$, $N = rad(V_2 \otimes V_p)$, and $\frac{V_2 \otimes V_p}{N} \cong V_{p-1}$, from theorem (0.4.10) and corollary (0.4.11).
If N is the unique maximal submodule of $V_2 \otimes V_p$ and V is the unique minimal submodule, then we have $\frac{V_2 \otimes V_p}{V} \cong V_{p+1}$.

We need to prove that $\frac{N}{V} \cong V_2$: Let $f : V_2 \longrightarrow V_{p+1}$, also X & Y are sent to X^p & Y^p, respectively, where $F[X,Y] \longrightarrow F[X,Y]$ is an FG-module homomorphism, one-to-one and onto, then f is an FG-module isomorphism. Then, V_{p+1} has a submodule isomorphic with V_2. Hence,

$$\frac{V_2 \otimes V_p}{V} \cong V_2 \quad \& \quad \frac{V_2 \otimes V_p}{N} \cong V_2;$$

so, $\frac{N}{V} \cong V_2$.

Then, the structure of $V_2 \otimes V_p$ is

$$\begin{array}{c} V_{p-1} \\ | \\ V_2 \\ | \\ V_{p-1} \end{array}.$$

Finally, if $1 < n < p-1$, P_n is uniserial, then from theorem (0.5.5), $\frac{rad(P_n)}{soc(P_n)} \cong V_{p+1-n}$ and from theorem (0.5.4), then $\frac{rad(P_n)}{soc(P_n)} \cong V_{p-1-n}$. Now, at $1 < n < p-1$, thus

$$\frac{rad(P_n)}{soc(P_n)} \cong V_{p+1-n} \oplus V_{p-1-n}$$

if and only if $\dim P_n = n + n + (p+1-n) + (p-1-n) = 2p$.

Then, the structure of P_n if $1 < n < p-1$ is

$$\begin{tikzcd}
 & V_n \arrow[dl, dash] \arrow[dr, dash] & \\
V_{p-1-n} \arrow[dr, dash] & & V_{p+1-n} \arrow[dl, dash] \\
 & V_n &
\end{tikzcd}$$

Now, if P_{p-2} is projective. We study P_{p-2} by examining $V_2 \otimes P_{p-1}$.

$V_2 \otimes P_{p-1}$ is projective, because P_{p-1} is projective, and by theorem (0.5.6) then $V_2 \otimes P_{p-1}$ is projective.

Then, $V_2 \otimes P_{p-1}$ has composition series $V_2 \otimes V_{p-1}$, $V_2 \otimes V_2$, and $V_2 \otimes V_{p-1}$ by theorem (0.5.5); that is isomorphic with $V_{p-2} \oplus V_p$, $V_1 \oplus V_3$, and $V_{p-2} \oplus V_p$ by theorem (0.5.7).
Since V_p is projective and V_{p-2} is homomorphic image, then $V_2 \otimes P_{p-1}$ has a direct summand isomorphic with $P_{p-2} \oplus V_p \oplus V_p$, but at $p = 3$ then $P_1 \oplus V_3 \oplus V_3 \oplus V_3$.
Then, P_{p-2} has two composition factors, where the radical and socle are V_{p-2}. From theorem (0.5.4) and theorem (0.5.5), we deduce that $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ has V_1 and V_3 are composition factor, but if $p = 3$, then just V_1; so $P_{p-2} = P_1$.

We need just see that $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ is isomorphic with $V_1 \oplus V_3$.
We need to prove $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ is semisimple; we prove it by contradiction. If it is not semisimple, then it must be uniserial (i.e. it has unique composition series), where it has just two composition factors.
Let $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ be not isomorphic with its dual (more details on dual are given in [2], §6, p.38). When taking dual the composition factor, we will exchange their order, but P_{p-2} is isomorphic its dual, because P^*_{p-2} is the indecomposable projective corresponding with $soc(P_{p-2})^* \cong V^*_{p-2} \cong V_{p-2}$ also, $Rad(P^*_{p-2}) \cong V^*_{p-2} \cong V_{p-2}$.
From the relations between radicals, socles, and duality imply that $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ is isomorphic with $\dfrac{rad(P^*_{p-2})}{soc(P_{p-2})^*}$. This is contradiction; hence $\dfrac{rad(P_{p-2})}{soc(P_{p-2})}$ is semisimple.

Next, we have established the structure of P_{p-2}.
Let $P_n \ \ \forall 2 < n < p-1$, be projective the composition series is $V_2 \otimes V_n$, $V_2 \otimes (V_{p+1-n} \oplus V_{p-1-n})$, and $V_2 \otimes V_n$. Therefore, from theorem (0.5.4), theorem (0.5.5), and theorem (0.5.7), hence that is isomorphic with $V_{n-1} \oplus V_{n+1}, V_{p-n} \oplus V_{p-n+2} \oplus V_{p-2-n} \oplus V_{p-n}$ and $V_{n-1} \oplus V_{n+1}$. But, if $n = p-2$, then the term V_{p-n-2} should be deleted, so

$$\frac{rad(P_{p-2})}{soc(P_{p-2})} \cong V_1 \oplus V_3.$$

Then, the composition series of P_{p-2} is $V_2 \otimes V_{p-2}$, $V_2 \otimes (V_3 \oplus V_1)$, and $V_2 \otimes V_{p-2}$; that is isomorphic with $V_{p-3} \oplus V_{p-1}, V_2 \oplus V_2 \oplus V_4$ and $V_{p-3} \oplus V_{p-1}$.

Thus, the structure of P_{p-2} is

$$\begin{tikzcd} & V_{p-2} \arrow[dl, dash] \arrow[dr, dash] & \\ V_1 \arrow[dr, dash] & & V_3 \arrow[dl, dash] \\ & V_{p-2} & \end{tikzcd}.$$

The following theorem summarizes the structure of the projective indecomposable $FSL(2,p)$-modules.

THEOREM 5.2.1. *([2], p.48). The projective indecomposable $FSL(2,p)$-modules have the following structures:*

$$\begin{array}{cccc} P_1 & P_n;\ 1 < n < p-1 & P_{p-1}; p > 2 & \\ \begin{tikzcd} V_1 \arrow[d, dash] \\ V_{p-2} \arrow[d, dash] \\ V_1 \end{tikzcd}, & \begin{tikzcd} & V_n \arrow[dl, dash] \arrow[dr, dash] & \\ V_{p-1-n} \arrow[dr, dash] & & V_{p+1-n} \arrow[dl, dash] \\ & V_n & \end{tikzcd}, & \begin{tikzcd} V_{p-1} \arrow[d, dash] \\ V_2 \arrow[d, dash] \\ V_{p-1} \end{tikzcd}, & \textit{and } P_p. \end{array}$$

P_1 and P_{p-1} are uniserial, while $P_p = St_G$ is simple projective.

5.3 THE INDECOMPOSABLE $FSL(2,p)$-MODULES

It is known (by counting elements) that the group $G = SL(2,p)$ is of order $p(p^2-1)$, and so has a (cyclic) Sylow p-subgroup of order p, namely $U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \forall \lambda \in \mathbb{F}_p \right\}$. It follows, by Higman's criterion theorem(0.3.7), that $FSL(2,p)$ is of finite representation type. In fact, $SL(2,p)$ is the only finite group of Lie type with this property [11]. Then in this section, we describe the complete set of indecomposable $FSL(2,p)$-modules as stated in the paper of D. Craven [4], p.54.

First, we introduce some important concepts for the indecomposable FG-module.

DEFINITION 5.3.1. *([18], p.104). Let M be an indecomposable FG-module, and let set $vx(M) = \{V \leq G : M$ is a V-projective $\}$; i.e. $vx(M) = \{V \leq G : M_V$ is a projective indecomposable FV-module$\}$. Then the minimal elements (by order) in $vx(M)$ are called the vertices of M.*

DEFINITION 5.3.2. *([9], p.339). Let M be an indecomposable FG-module with vertex V, and let W be an indecomposable FV-module. Then W is a source of M if $M|W^G$.*

REMARK. *([11], p.72). Let* $\Lambda = FG$ *be a group algebra, let* V *be a* p*-subgroup of* G*, let* W *be an indecomposable* FV*-module, and let* M *be an indecomposable* Λ*-module, which is a direct summand of a module induced from* V *to* G*; i.e.* $M|ind_V^G W$*. If* V *is minimal subgroup, then* V *is vertex of* M*, while* W *is a source of* M*.*

THEOREM 5.3.3. *Let* M *be an indecomposable* FG*-module, and let* U *be a Sylow* p*-subgroup of* G*. Then* $|U : vx(M)|$ *divides* $dim M$*.*

THEOREM 5.3.4. *Let* $\Lambda = FSL(2,p)$ *be a group algebra in characteristic* p*, where* p *is all prime numbers. Then any non-projective indecomposable* Λ*-module and simple* Λ*-module have vertex Sylow* p*-subgroup* U*, and all projective indecomposable* Λ*-modules* $P_1, P_2, \dots, P_p$ *have vertex trivial subgroup* I*.*

Proof. First, let $U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \forall \lambda \in \mathbb{F}_p \right\}$ be the Sylow p-subgroup, where $|U| = p$, let V_n be a simple Λ-module, where $n = \{1, 2, \dots, p-1\}$ and let M be a non-projective indecomposable Λ-module. Since $\dim V_n = n$ for all $n = \{1, 2, \dots, p-1\}$, and p does not divide $\dim M = x$. Then from theorem(5.3.3), $|U : vx(V_n)|$ divides $\dim V_n$; i.e. $\dfrac{p}{|vx(V_n)|}$ divides n, so $vx(V_n)$ must be U.

Thus, the vertex of all simple Λ-modules is the Sylow p-subgroup U.

Similarly, from theorem (5.3.3), $|U : vx(M)|$ divides $\dim M$; i.e. $\dfrac{p}{|vx(M)|}$ divides x, so $vx(M)$ must be U.
Thus, The vertex of all non-projective indecomposable Λ-modules is the Sylow p-subgroup U.

Second, let P_i be projective indecomposable Λ-modules, where $i = \{1, 2, \dots, p\}$. P_i are a direct summand of Λ; i.e. $P_i|ind_I^G F$, where I is the trivial subgroup, then the vertex of P_i for all i, are the trivial subgroup I. $\square$

Of the most important concepts in an indecomposable A-module is almost split sequence. So, we introduce the definition as follows:

DEFINITION 5.3.5. *(Auslander-Reiten) ([18], p.151). Let* A *be a finite dimension algebra over a field* F *of characteristic* p*. An almost split sequence is a short exact non-split sequence of* A*-modules.*

$$0 \to X \xrightarrow{i} Y \xrightarrow{f} Z \to 0.$$

Also, it is called terminates in Z*, in which* X *and* Z *are both indecomposable such that there exists* A*-module homomorphism* $\rho : W \to Z$ *is not a split epimorphism.*

The following theorem shows that, in group algebra FG, if the characteristic p divides the order of group G, then there is almost split sequence terminating.

THEOREM 5.3.6. *(Auslander-Reiten) ([18], p.151). Let* F *be a field of characteristic* p*, and let* G *be a finite group, where* p *divides the order of* G*. If* M *is an indecomposable* FG*-module, then there exists an almost split sequence terminating in* M*.*

The following theorem shows that, there is a one-to-one correspondence between the isomorphism classes of all non-projective indecomposable FG-module and the isomorphism classes of all non-projective indecomposable $FN_G(U)$-module.

THEOREM 5.3.7. *(The Green Correspondence) ([2], Section 11). There is a one-to-one correspondence between isomorphism classes of indecomposable FG-modules with vertex in ε and isomorphism classes of indecomposable FN-modules with vertex in ε; where N is normalizer of Sylow p-subgroup. If M and V are such modules for G and N, respectively, which correspond then M and V have the same vertex and*

$$M_N \cong V \oplus Y,$$

$$V^G \cong M \oplus X.$$

Where Y is a projective FN-module and X is a projective FG-module, then there exists a bijection

$$M \longleftrightarrow V.$$

We fix some notation;
Let U be a p-subgroup of G, and let N be a subgroup containing normalizer of Sylow p-subgroup $N = N_G(U) = \{x \in G : xUx^{-1} = U\} \leq G$. If L and B are subgroups of G, where $(L \subseteq B)$.
We fix some collections of p-subgroups of G.

$$\aleph = \{sUs^{-1} \cap U : s \in G, s \notin N\},$$
$$\Im = \{sUs^{-1} \cap N : s \in G, s \notin N\},$$
$$\varepsilon = \{B : B \subset U, B \notin \aleph\}.$$

We know that the group algebra FG has p simple FG-modules $V_1, V_2, \ldots, V_p$, where V_n has dimension n; $\forall n = 1, 2, \ldots, p$. Also, we have the structure of the projective indecomposable $FSL(2,p)$-modules as stated in theorem (5.2.1):

P_1 $\qquad$ $P_n;\ 1 < n < p-1$ $\qquad$ $P_{p-1}; p > 2$

$$\begin{tikzcd}[arrows=dash] V_1 \arrow[d] \\ V_{p-2} \arrow[d] \\ V_1 \end{tikzcd}, \qquad \begin{tikzcd}[arrows=dash] & V_n \arrow[dl] \arrow[dr] & \\ V_{p-1-n} \arrow[dr] & & V_{p+1-n} \arrow[dl] \\ & V_n & \end{tikzcd}, \qquad \begin{tikzcd}[arrows=dash] V_{p-1} \arrow[d] \\ V_2 \arrow[d] \\ V_{p-1} \end{tikzcd}, \qquad V_p \cong P_p.$$

Now, we construct all non-simple, non-projective indecomposable modules for group algebra $FSL(2,p)$ in characteristic $p \geq 5$ by using the projective indecomposable modules:

We cannot use the following theorem: Any indecomposable FG-module is a homomorphic image of projective indecomposable FG-module (theorem 0.3.11), because: $U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \forall \lambda \in \mathbb{F}_p \right\}$ is not normal in $G = SL(2,p)$, where the Sylow p-subgroup must be cyclic and normal in G as stated in ([2], p.42).

So, we use Green correspondence theorem (5.3.7) in which there is a one-to-one correspondence between the isomorphism classes of all non-projective indecomposable FG-module and the isomorphism classes of all non-projective indecomposable $FN_G(U)$-module, where

$$N_G(U) = \left\{ \begin{pmatrix} x^{-1} & \lambda \\ 0 & x \end{pmatrix} : \forall \lambda, x \in \mathbb{F}_p \right\}$$

is normalizer of the Sylow p-subgroup U of G. Then U is normal and cyclic Sylow p-subgroup in $N_G(U)$.

Hence, **the Green Correspondence in theorem (5.3.7)** shows that the number of indecomposable FG-modules is equal to the number of indecomposable $FN_G(U)$-modules. It turns out that there are $p^2 - p + 1$ isomorphism classes of indecomposable $FSL(2, p)$-modules: $p - 1$ of them is simple non-projective, one is simple projective, $p - 1$ projective non-simple, and $(p - 1)(p - 2)$ are non-simple non-projective indecomposable $FSL(2, p)$-modules.

We want to completely describe the indecomposable FG-module, since U is normal and cyclic Sylow p-subgroup in $N_G(U)$, then this is very much like the case for cyclic group as stated in ([2], p.35).

Thus, the non-simple, non-projective indecomposable FG-modules are

$$\begin{tikzcd} & V_i \arrow[dl, dash] \arrow[dr, dash] & \\ V_{p-1-i} & & V_{p+1-i} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V_{p-1-i} \arrow[dr, dash] & & V_{p+1-i} \arrow[dl, dash] \\ & V_i & \end{tikzcd} \quad .$$

We can remove the socles of the two projectives, where they are different from each other, and we take their direct sum and then quotient out by a diagonal.
This process certainly produces indecomposable module. But, if we take the direct sum of two copies of projective is not indecomposable module, because it becomes split extension.

One can continue this process until one constructs an indecomposable module M with all (non-projective) simple module, where this constructs all non-projective indecomposable modules for $FSL(2, p)$, and the non-simple indecomposable subquotients of the module M are one-to-one correspondence with connected subdiagrams of the diagram with at least one edge.

We know that

$$Ext_{FG}(V_i, V_j) \cong Hom_{FG}(rad(P_i), V_j),$$

where P_i is the projective cover of V_i as stated in ([11], p.116). Then, from projective indecomposable FG-module P_1; this gives dim $Ext_{FG}(V_1, V_{p-2}) = 1$.
The projective indecomposable FG-module P_{p-1}; $p > 2$, this gives dim $Ext_{FG}(V_{p-1}, V_2) = 1$.
Also, the projective indecomposable FG-module P_n; $1 < n < p - 1$, this gives dim $Ext_{FG}(V_n, V_{p+1-n}) = 1$ & dim $Ext_{FG}(V_n, V_{p-1-n}) = 1$.
So, we get the following proposition:

PROPOSITION 5.3.8. *([11], p.117). Let $FSL(2,p)$ be a group algebra in characteristic p, where p is all prime numbers. Then $Ext_{FG}(V_i, V_j) \neq 0$ (i.e. the short exact sequence is non-split extension of V_i by V_j from definition (0.5.3)) for all simple FG-modules V_i, V_j are given as follows:*

1. *If $p = 2$, then dim $Ext_{FG}(V_1, V_1) = 1$;*

2. *If p is odd, then dim $Ext_{FG}(V_n, V_{p+1-n}) = 1$ and dim $Ext_{FG}(V_n, V_{p-1-n}) = 1$ for $1 < n < p-1$, while $Ext_{FG}(V_1, V_{p-2})$ and $Ext_{FG}(V_{p-1}, V_2)$ are 1-dimension.*

Then, we have:

PROPOSITION 5.3.9. *([4], p.56). Let $FSL(2,p)$ be a group algebra in characteristic p, where $G = SL(2,p)$, and let M be an indecomposable FG-module.*

1. *If M has one socle layer, then M is simple. There are p simple FG-modules.*

2. *If M has three socle layers, then $M = P_i; 1 \leq i \leq p-1$ are projective indecomposable FG-module.*

3. *If M has two socle layers, then the socle of M consists of simple module of dimension $n, n+2, \ldots, j (n \leqslant j)$, and the top consists of modules of dimension $p - j + \varepsilon, p - j + \varepsilon + 2, \ldots, p - n + \delta$, where $\varepsilon, \delta = \pm 1$. There are $(p-1)(p-2)$ indecomposable module.*

EXAMPLE 5.3.10. *Let F be a field of characteristic 5, and let $SL(2,5)$ be a special linear group. Hence, the number of all indecomposable $FSL(2,5)$-modules are $p^2 - p + 1 = 21$; $p = 5$.*

***The simple** $FSL(2,5)$**-modules** are V_1, V_2, V_3, V_4, and $V_5 \cong P_5$ is Steinberg module;*
The projective indecomposable** $FSL(2,5)$**-modules have the following structures:

$$
\begin{tikzcd}[column sep=small]
 & P_1 & & P_2 & & & P_3 & & P_4 \\
 & V_1 \arrow[d, dash] & & V_2 \arrow[dl, dash] \arrow[dr, dash] & & & V_3 \arrow[dl, dash] \arrow[dr, dash] & & V_4 \arrow[d, dash] \\
 & V_3 \arrow[d, dash] & V_2 \arrow[dr, dash] & & V_4 \arrow[dl, dash] & V_1 \arrow[dr, dash] & & V_3 \arrow[dl, dash] & V_2 \arrow[d, dash] \\
 & V_1 , & & V_2 , & & & V_3 , & & V_4 ,
\end{tikzcd}
\quad \text{and } V_5 \cong P_5.
$$

The non-simple, non-projective indecomposable** $FSL(2,5)$**-modules are

$$
\begin{tikzcd}[column sep=small]
 & V_3 \arrow[dl, dash] \arrow[dr, dash] & & V_3 \arrow[dr, dash] & & V_1 \arrow[dl, dash] & V_1 \arrow[d, dash] & V_3 \arrow[d, dash] & V_3 \arrow[d, dash] \\
V_1 & & V_3 , & & V_3 , & & V_3 , & V_1 , & V_3 ,
\end{tikzcd}
$$

$$\begin{tikzcd}
& V_3 \arrow[dl, dash] \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\
V_1 & & V_3 &
\end{tikzcd},$$

$$\begin{tikzcd}
& V_2 \arrow[dl, dash] \arrow[dr, dash] & \\
V_2 & & V_4
\end{tikzcd},\quad
\begin{tikzcd}
V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \\
& V_2 &
\end{tikzcd},\quad
\begin{tikzcd}
V_2 \arrow[d, dash] \\
V_2
\end{tikzcd},\quad
\begin{tikzcd}
V_4 \arrow[d, dash] \\
V_2
\end{tikzcd},\quad
\begin{tikzcd}
V_2 \arrow[d, dash] \\
V_4
\end{tikzcd},$$

and

$$\begin{tikzcd}
V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \arrow[dr, dash] & \\
& V_2 & & V_4
\end{tikzcd}.$$

From lemma (5.1.2), then the odd-dimensional simple modules V_1, V_3, *and* $V_5 \cong P_5$ *are modules for* $PSL(2,5)$. *Also, it contains indecomposable* $FPSL(2,5)$*-modules* P_1, P_3,

$$\begin{tikzcd}
& V_3 \arrow[dl, dash] \arrow[dr, dash] & \\
V_1 & & V_3
\end{tikzcd},\quad
\begin{tikzcd}
V_3 \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\
& V_3 &
\end{tikzcd},\quad
\begin{tikzcd}
V_1 \arrow[d, dash] \\
V_3
\end{tikzcd},\quad
\begin{tikzcd}
V_3 \arrow[d, dash] \\
V_1
\end{tikzcd},\quad
\begin{tikzcd}
V_3 \arrow[d, dash] \\
V_3
\end{tikzcd},$$

and

$$\begin{tikzcd}
& V_3 \arrow[dl, dash] \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\
V_1 & & V_3 &
\end{tikzcd}.$$

The even-dimensional simple modules V_2 *and* V_4 *are faithful modules for* $SL(2,5)$. *Also, it contains indecomposable* $FSL(2,5)$*-modules* P_2, P_4,

$$\begin{tikzcd}
& V_2 \arrow[dl, dash] \arrow[dr, dash] & \\
V_2 & & V_4
\end{tikzcd},\quad
\begin{tikzcd}
V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \\
& V_2 &
\end{tikzcd},\quad
\begin{tikzcd}
V_2 \arrow[d, dash] \\
V_2
\end{tikzcd},\quad
\begin{tikzcd}
V_4 \arrow[d, dash] \\
V_2
\end{tikzcd},\quad
\begin{tikzcd}
V_2 \arrow[d, dash] \\
V_4
\end{tikzcd},$$

and

$$\begin{tikzcd}
V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \arrow[dr, dash] & \\
& V_2 & & V_4
\end{tikzcd}.$$

Now, we find the vertices of indecomposable $FSL(2,5)$-modules; the Sylow 5-subgroup is $U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \forall \lambda \in \mathbb{F}_5 \right\}$, *where* $|U| = 5$.
The vertex of V_1, V_2, V_3 and V_4 is Sylow 5-subgroup U, also the vertex of non-simple, non-projective indecomposable $FSL(2,5)$-modules is U from theorem (5.3.4).
The vertex of each projective indecomposable $FSL(2,5)$-modules P_1, P_2, P_3, P_4, and P_5 is the trivial subgroup I from theorem (5.3.4).

5.4 BLOCK THEORY OF $FSL(2,p)$-MODULES

In this section, in [2, 9], we find that the group algebra $\Lambda = FSL(2,p)$ in characteristic odd prime number p has three blocks by using Brauer Correspondent.

DEFINITION 5.4.1. *([2], p.102). Let U be a p-subgroup of G, and let $N_G(U)$ be normalizer of G. Then there is a one-to-one correspondence between the blocks of $N_G(U)$ with defect group U and the blocks of G with defect group U given by letting the block b of $N_G(U)$ correspond with the block b^G of G. Thus, the one-to-one correspondence is called Brauer correspondent.*

The following theorem shows that, The group algebra $FSL(2,p)$ in characteristic odd prime number p has blocks $B_1, B_2,$ and B_3.

THEOREM 5.4.2. *([9], p.469). Let F be a field of characteristic odd prime number p, and let $G = SL(2,p)$ be a special linear group. Then the group algebra FG has three blocks $B_1, B_2,$ and B_3, where B_1 contains all odd-dimensional simple FG-modules except the Steinberg module V_p; B_2 contains all even-dimensional simple FG-modules, and the block of defect zero B_3 contains the Steinberg module V_p, where B_1 and B_2 are defect 1.*

Proof. We know that

$$N_G(U) = \left\{ \begin{pmatrix} x^{-1} & \lambda \\ 0 & x \end{pmatrix} : \forall \lambda \in \mathbb{F}_p \wedge x \in \mathbb{F}_p \right\}.$$

The group algebra $FN_G(U)$ has no blocks of defect zero; then from Brauer correspondent, the blocks of $FN_G(U)$ with defect group U are one-to-one correspondent with the blocks of FG with defect group U; where U is defect 1.

Since all odd-dimensional simple modules are modules for projective linear group $PSL(2,p)$, and the even-dimensional simple modules are faithful module for $SL(2,p)$ from lemma (5.1.2), then $FN_G(U)$ has two blocks of defect 1, hence FG has two blocks of defect 1. Let B_1, B_2 be blocks of defect 1.

From theorem (1.2.4), Let X, Y be two simple Λ-modules. Then X, Y lie in the same block (i.e. $X \underset{\Lambda}{\approx} Y$) if and only if there is a sequence from projective indecomposable modules $P_j = P_1, P_2, \dots, P_t = P_k$ corresponding the simple Λ-modules such that

$$(P_i, P_{i+1})_\Lambda \neq 0 \quad or \quad (P_{i+1}, P_i)_\Lambda \neq 0.$$

Claim that $V_i, V_{i'} \in B_1$, $\forall i, i'$ are odd numbers.
For example at $p = 5$; let V_3, V_1 be corresponding the projective indecomposable Λ-modules P_3, P_1 respectively. Then there is Λ-module homomorphism between P_3 & P_1; i.e. $(P_3, P_1)_\Lambda \neq 0$ as follows:

$$P_3 \to \begin{array}{ccc} & V_3 & \\ V_1 & & V_3 \end{array} \to \begin{array}{c} V_3 \\ V_3 \end{array} \to V_3 \to \begin{array}{c} V_1 \\ V_3 \end{array} \to V_1 \to \begin{array}{c} V_3 \\ V_1 \end{array} \to P_1.$$

Thus, $V_3 \underset{\Lambda}{\approx} V_1$.

Now, let V_i, $V_{i'}$, $V_{i''}$ be odd-dimensional simple Λ-modules, where i, i', i'' are odd numbers, and let

$$\begin{array}{c} P_1 \\ \\ V_1 \\ V_{p-2} \\ V_1 \end{array}, \qquad \begin{array}{ccc} & P_i;\ 1 < i < p-1 & \\ & & \\ & V_i & \\ V_{p-1-i} & & V_{p+1-i} \\ & V_i & \end{array}$$

be two projective indecomposable Λ-modules. Then $V_i \underset{\Lambda}{\approx} V_1$, because the projective covers P_1, P_i of V_1, V_i respectively are connected by a series of Λ-module homomorphisms as follows:

$$P_i \to \begin{array}{ccc} & V_i & \\ V_{p-1-i} & & V_{p+1-i} \end{array} \to V_i \to \begin{array}{c} V_1 \\ V_{p-2} \end{array} \to V_1 \to \begin{array}{c} V_{p-2} \\ V_1 \end{array} \to P_1.$$

Also, $V_i \underset{\Lambda}{\approx} V_{i'}$, because the projective covers P_i, $P_{i'}$ of V_i, $V_{i'}$ respectively are connected by a series of Λ-module homomorphisms as follows:

$$P_i \to \begin{array}{ccc} & V_i & \\ V_{p-1-i} & & V_{p+1-i} \end{array} \to V_i \to \begin{array}{c} V_{i'} \\ V_i \end{array} \to \begin{array}{ccc} & V_{i'} & \\ V_i & & V_{i''} \end{array} \to$$

$$V_{i'} \to \begin{array}{ccc} V_i & & V_{i''} \\ & V_{i'} & \end{array} \to P_{i'}.$$

Hence, $(P_i, P_{i'})_\Lambda \neq 0$ or $(P_{i'}, P_i)_\Lambda \neq 0$ for all i, i' are odd numbers; so $V_i \underset{\Lambda}{\approx} V_{i'}$.

Since the prime number p is odd number, then we must exclude the Steinberg module $V_p \cong P_p$; i.e. V_p does not belong to B_1, because there is no Λ-module homomorphism between P_p and P_i for all odd numbers i; i.e. $(P_p, P_i)_\Lambda = 0$ and $(P_i, P_p)_\Lambda = 0$.

Thus, B_1 contains all odd-dimensional simple modules, and projective modules, also non-simple, non-projective, indecomposable $FPSL(2,p)$-modules except the Steinberg module V_p.

Claim that $V_j, V_{j'} \in B_2, \forall j, j'$ are even numbers.
For example at $p = 5$; let V_2, V_4 be two simple Λ-modules corresponding the projective indecomposable Λ-modules P_2, P_4 respectively. Then there is Λ-module homomorphism between P_2, P_4; i.e. $(P_2, P_4)_\Lambda \neq 0$ as follows:

$$
\begin{tikzcd}[arrows=dash]
 & & V_2 \arrow[dl] \arrow[dr] & & & V_2 \arrow[d] & & V_4 \arrow[d] & & V_2 \arrow[d] & \\
P_2 \rightarrow & V_2 & & V_4 & \rightarrow & V_2 & \rightarrow V_2 \rightarrow & V_2 & \rightarrow V_4 \rightarrow & V_4 & \rightarrow P_4.
\end{tikzcd}
$$

Thus, $V_2 \underset{\Lambda}{\approx} V_4$.

Now, let V_j, $V_{j'}$, $V_{j''}$ be even-dimensional simple Λ-modules, where j, j', j'' are even numbers, and let

$$
P_{p-1}; p > 2 \qquad\qquad P_j;\ 1 < j < p-1
$$

$$
\begin{tikzcd}[arrows=dash]
V_{p-1} \arrow[d] & & V_j \arrow[dl] \arrow[dr] & \\
V_2 \arrow[d] & V_{p-1-j} \arrow[dr] & & V_{p+1-j} \arrow[dl] \\
V_{p-1}, & & V_j &
\end{tikzcd}
$$

be two projective indecomposable Λ-modules. Then $V_j \underset{\Lambda}{\approx} V_{p-1}$, because the projective covers P_j, P_{p-1} of V_j, V_{p-1} respectively are connected by a series of Λ-module homomorphisms as follows:

$$
\begin{tikzcd}[arrows=dash]
 & & V_j \arrow[dl] \arrow[dr] & & & V_{p-1} \arrow[d] & & V_2 \arrow[d] & \\
P_j \rightarrow & V_{p-1-j} & & V_{p+1-j} & \rightarrow V_j \rightarrow & V_2 & \rightarrow V_{p-1} \rightarrow & V_{p-1} & \rightarrow P_{p-1}.
\end{tikzcd}
$$

Also, $V_j \underset{\Lambda}{\approx} V_{j'}$, because the projective covers P_j, $P_{j'}$ of V_j, $V_{j'}$ respectively are connected by a series of Λ-module homomorphisms as follows:

$$P_j \to \begin{array}{ccc} & V_j & \\ \diagup & & \diagdown \\ V_{p-1-j} & & V_{p+1-j} \end{array} \to V_j \to \begin{array}{c} V_{j`} \\ | \\ V_j \end{array} \to \begin{array}{ccc} & V_{j`} & \\ \diagup & & \diagdown \\ V_j & & V_{j``} \end{array} \to$$

$$V_{j`} \to \begin{array}{ccc} V_j & & V_{j``} \\ & \diagdown \quad \diagup & \\ & V_{j`} & \end{array} \to P_{j`}.$$

Hence, $(P_j, P_{j`})_\Lambda \neq 0$ or $(P_{j`}, P_j)_\Lambda \neq 0$ for all $j, j`$ are even numbers; so $V_j \underset{\Lambda}{\approx} V_{j`}$.

Thus, B_2 contains all simple modules that they have even-dimensional, and projective modules, also non-simple, non-projective, indecomposable $FSL(2,p)$-modules, where the even-dimensional are faithful simple module for $FSL(2,p)$-module.

Also, Λ has the block of defect zero contains the Steinberg module; i.e. $B_3 = \{V_p \cong P_p\}$.

$\square$

EXAMPLE 5.4.3. ***From example(5.3.10)****; there are three blocks of $FSL(2,5)$ as follows:*

The block B_1 contains

$$V_1, V_3, \begin{array}{c} V_1 \\ | \\ V_3 \\ | \\ V_1 \end{array}, \begin{array}{ccc} & V_3 & \\ \diagup & & \diagdown \\ V_1 & & V_3 \\ \diagdown & & \diagup \\ & V_3 & \end{array}, \begin{array}{ccc} & V_3 & \\ \diagup & & \diagdown \\ V_1 & & V_3 \end{array}, \begin{array}{ccc} V_3 & & V_1 \\ & \diagdown \quad \diagup & \\ & V_3 & \end{array},$$

$$\begin{array}{c} V_1 \\ | \\ V_3 \end{array}, \begin{array}{c} V_3 \\ | \\ V_1 \end{array}, \begin{array}{c} V_3 \\ | \\ V_3 \end{array}, \text{ and } \begin{array}{ccccc} & V_3 & & V_1 & \\ \diagup & & \diagdown \quad \diagup & & \\ V_1 & & V_3 & & \end{array}.$$

The block B_2 contains

$$V_2, V_4, \begin{array}{ccc} & V_2 & \\ \diagup & & \diagdown \\ V_2 & & V_4 \\ \diagdown & & \diagup \\ & V_2 & \end{array}, \begin{array}{c} V_4 \\ | \\ V_2 \\ | \\ V_4 \end{array}, \begin{array}{ccc} & V_2 & \\ \diagup & & \diagdown \\ V_2 & & V_4 \end{array}, \begin{array}{ccc} V_4 & & V_2 \\ & \diagdown \quad \diagup & \\ & V_2 & \end{array},$$

$$\begin{array}{c} V_2 \\ | \\ V_2 \end{array}, \quad \begin{array}{c} V_4 \\ | \\ V_2 \end{array}, \quad \begin{array}{c} V_2 \\ | \\ V_4 \end{array}, \quad \text{and} \quad \begin{tikzcd}[arrows=dash] V_4 \arrow[dr] & & V_2 \arrow[dl] \arrow[dr] & \\ & V_2 & & V_4 \end{tikzcd}.$$

The block B_3 ***contains the Steinberg module*** $B_3 = \{V_5 \cong P_5\}$.

5.5 THE PSEUDOBLOCKS OF $FSL(2, p)$

Here, we determine the pseudoblocks of the group algebra $\Lambda = FSL(2, p)$ in characteristic prime p, and then compare the two notions "blocks" and "pseudoblocks" in group algebra Λ.

THEOREM 5.5.1. *[1]. Let F be a field of characteristic p, and let $G = SL(2, p)$ be a special linear group, where p is odd prime number. Then the group algebra $\Lambda = FG$ has three pseudoblocks. Moreover, for the group algebra Λ in characteristic prime p, the block and pseudoblock notions coincide.*

Proof. From theorem (5.4.2), there exist three blocks of Λ in characteristic odd prime number p, which are B_1, B_2, and B_3.

First: The block B_3 (which contains the Steinberg module $V_p \cong P_p$) is clearly pseudoblock.

Second: Since B_1 contains all simple modules, projective modules, and indecomposable $F(PSL(2, p))$-modules except the Steinberg module V_p; i.e. B_1 contains all odd-dimensional simple Λ-modules except V_p.

Let P_m, P_i be projective indecomposable Λ-modules, let V_m, V_i be simple Λ-modules; for all $m, i \in \{1, 3, \ldots, p-2\}$, and let $M_{i^\backprime}$ be non-simple, non-projective, indecomposable Λ-modules, where $M_{i^\backprime}$ has two socle layers; $i^\backprime = \{1, 2, \ldots, r\}$; in which P_m, V_m, P_i, V_i and $M_{i^\backprime}$ in B_1 for all $m, i, i^\backprime$. Let

$$P_i = \begin{tikzcd}[arrows=dash] & V_i \arrow[dl] \arrow[dr] & \\ V_{p-1-i} \arrow[dr] & & V_{p+1-i} \arrow[dl] \\ & V_i & \end{tikzcd}, \quad P_m = \begin{tikzcd}[arrows=dash] & V_m \arrow[dl] \arrow[dr] & \\ V_{p-1-m} \arrow[dr] & & V_{p+1-m} \arrow[dl] \\ & V_m & \end{tikzcd},$$

$$M_1 = \begin{tikzcd}[arrows=dash] & V_i \arrow[dl] \arrow[dr] & \\ V_{p-1-i} & & V_{p+1-i} \end{tikzcd}, \quad M_2 = \begin{array}{c} V_{p+1-i} \\ | \\ V_i \end{array},$$

$$M_3 = \begin{tikzcd}[column sep=small] V_{p+1-m} \arrow[dr, dash] & & V_{p-1-m} \arrow[dl, dash] \\ & V_m & \end{tikzcd}, \quad M_4 = \begin{tikzcd}[column sep=small] & V_{p+1-m} \arrow[dl, dash] \arrow[dr, dash] & & V_{p-1-m} \arrow[dl, dash] \\ V_i & & V_m & \end{tikzcd},$$

$$M_5 = \begin{tikzcd}[column sep=small] & V_m \arrow[dl, dash] \arrow[dr, dash] & \\ V_{p-1-m} & & V_{p+1-m} \end{tikzcd}, \quad M_6 = \begin{tikzcd}[column sep=small] V_{p+1-i} \arrow[dr, dash] & & V_{p-1-i} \arrow[dl, dash] \\ & V_i & \end{tikzcd}.$$

Then, we have six cases as follows:

1. Let V_i, V_m be any two simple Λ-modules. Hence,
$$V_i \to M_2 \to V_m.$$
Then, all odd-dimensional simple Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

2. Let V_i, V_m be simple Λ-modules, and let P_i, P_m be projective indecomposable Λ-modules. Hence,
$$P_i \to M_1 \to V_i, \text{ and } V_m \to M_3 \to P_m.$$
Then, all odd-dimensional simple Λ-modules and all projective indecomposable Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

3. Let $M_{\grave{i}}; \grave{i} = \{1, 2, 3, 5\}$ be any non-simple, non-projective, indecomposable Λ-modules, and let V_i, V_m be any two simple Λ-modules. Hence,
$$M_1 \to V_i, \quad M_2 \to V_m, \quad M_3 \to P_m \to M_5 \to V_m.$$
Then, all odd-dimensional simple Λ-modules and all non-simple, non-projective, indecomposable Λ-modules $M_{\grave{i}}; \grave{i} = \{1, 2, \ldots, r\}$ are connected either ways by a sequence of Λ-module homomorphisms.

4. Let P_i, P_m be any two projective indecomposable Λ-modules. Hence,
$$P_i \to M_1 \to V_i \to M_2 \to V_{p+1-i} \to M_3 \to P_m.$$
Then, all projective indecomposable Λ-modules $P_m, \forall m = \{1, 3, \ldots, p-2\}$ are connected either ways by a sequence of Λ-module homomorphisms.

5. Let P_i, P_m be any two projective indecomposable Λ-modules, and let M_1, M_3, M_5, M_6 be non-simple, non-projective,indecomposable Λ-modules. Hence,
$$P_i \to M_1 \ \& \ P_m \to M_5.$$
Also,
$$M_6 \to P_i \ \& \ M_3 \to P_m.$$
Then, all projective indecomposable Λ-modules $P_m, \forall m = \{1, 3, \ldots, p-2\}$ and all non-simple, non-projective, indecomposable Λ-modules $M_{\grave{i}}; \grave{i} = \{1, 2, \ldots, r\}$ are connected either ways by a sequence of Λ-module homomorphisms.

6. Let $M_1, M_2, M_3, M_4, M_5, M_6$ be any non-simple, non-projective, indecomposable Λ-modules. Hence,

$$M_6 \to P_i \to M_1,$$
$$M_1 \to V_i \to M_2,$$
$$M_3 \to P_m \to M_5,$$

and

$$M_4 \to M_3.$$

Then, all non-simple, non-projective, indecomposable Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable Λ-modules in B_1 are connected either ways by a sequence of Λ-module homomorphisms as follows:

$$P_i \to M_{i^\backprime} \to V_i \to \ldots \leftarrow M^{\backprime}_{i^\backprime} \leftarrow V_m \leftarrow M^{\backprime\backprime}_{i^\backprime} \leftarrow P_m;$$

for all $i, m \in \{1, 3, 5, \ldots, p-2\}$ and $i^\backprime = \{1, 2, \ldots, r\}$.
i.e. from definition (1.1.1), $\forall X, Y \in B_1$; there is a sequence of indecomposable modules $X = X_1, X_2, \ldots, X_t = Y$ in B_1 such that for all $n \in \{1, 2, \ldots, t\}$ either

$$(X_n, X_{n+1})_\Lambda \neq 0 \quad \text{or} \quad (X_{n+1}, X_n)_\Lambda \neq 0.$$

Thus, The block B_1 does not split into union of pseudoblocks. So, B_1 is one pseudoblock.

<u>Third:</u> Similarly, since the block B_2 contains all even-dimensional simple Λ-modules. Let P_e, P_j be projective indecomposable Λ-modules, let V_e, V_j be simple Λ-modules; for all $j, e \in \{2, 4, \ldots, p-1\}$, and let $N_{j^\backprime}$ be non-simple, non-projective, indecomposable Λ-modules, where $N_{j^\backprime}$ has two socle layers; $j^\backprime = \{1, 2, \ldots, r\}$; in which P_e, P_j, V_e, V_j, and $N_{j^\backprime}$ in B_2 for all $e, j, j^\backprime$. Let

$$P_j = \begin{tikzcd}[every arrow/.append style={dash}] & V_j \arrow[dl] \arrow[dr] & \\ V_{p-1-j} \arrow[dr] & & V_{p+1-j} \arrow[dl] \\ & V_j & \end{tikzcd}, \quad P_e = \begin{tikzcd}[every arrow/.append style={dash}] & V_e \arrow[dl] \arrow[dr] & \\ V_{p-1-e} \arrow[dr] & & V_{p+1-e} \arrow[dl] \\ & V_e & \end{tikzcd},$$

$$N_1 = \begin{tikzcd}[every arrow/.append style={dash}] & V_j \arrow[dl] \arrow[dr] & \\ V_{p-1-j} & & V_{p+1-j} \end{tikzcd}, \quad N_2 = \begin{tikzcd}[every arrow/.append style={dash}] V_{p+1-j} \arrow[d] \\ V_j \end{tikzcd},$$

$$N_3 = \begin{tikzcd}[every arrow/.append style={dash}] V_{p+1-e} \arrow[dr] & & V_{p-1-e} \arrow[dl] \\ & V_e & \end{tikzcd}, \quad N_4 = \begin{tikzcd}[every arrow/.append style={dash}] & V_{p+1-e} \arrow[dl] \arrow[dr] & & V_{p-1-e} \arrow[dl] \\ V_j & & V_e & \end{tikzcd},$$

$$N_5 = \begin{tikzcd} & V_e \arrow[dl, dash] \arrow[dr, dash] & \\ V_{p-1-e} & & V_{p+1-e} \end{tikzcd}, \quad N_6 = \begin{tikzcd} V_{p+1-j} \arrow[dr, dash] & & V_{p-1-j} \arrow[dl, dash] \\ & V_j & \end{tikzcd}.$$

Then, we have six cases as follows:

1. Let V_j, V_e be any two simple Λ-modules. Hence,

$$V_j \to N_2 \to V_e.$$

Then, all even-dimensional simple Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

2. Let V_j, V_e be simple Λ-modules, and let P_j, P_e be projective indecomposable Λ-modules. Hence,

$$P_j \to N_1 \to V_j \text{ and } V_e \to N_3 \to P_e.$$

Then, all even-dimensional simple Λ-modules and all projective indecomposable Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

3. Let $N_{j^\backprime}; j^\backprime = \{1, 2, 3, 5\}$ be any non-simple, non-projective, indecomposable Λ-modules, and let V_j, V_e be any two simple Λ-modules. Hence,

$$N_1 \to V_j, \quad N_2 \to V_e, \quad N_3 \to P_e \to N_5 \to V_e.$$

Then, all even-dimensional simple Λ-modules and all non-simple, non-projective, indecomposable Λ-modules $N_{j^\backprime}; j^\backprime = \{1, 2, \ldots, r\}$ are connected either ways by a sequence of Λ-module homomorphisms.

4. Let P_j, P_e be any two projective indecomposable Λ-modules. Hence,

$$P_j \to N_1 \to V_j \to N_2 \to V_{p+1-j} \to N_3 \to P_e.$$

Then, all projective indecomposable Λ-modules $P_e, \forall e = \{2, 4, \ldots, p-1\}$ are connected either ways by a sequence of Λ-module homomorphisms.

5. Let P_j, P_e be any two projective indecomposable Λ-modules, and let N_1, N_3, N_5, N_6 be non-simple, non-projective,indecomposable Λ-modules. Hence,

$$P_j \to N_1 \;\&\; P_e \to N_5.$$

Also,

$$N_6 \to P_j \;\&\; N_3 \to P_e.$$

Then, all projective indecomposable Λ-modules $P_e; \forall e = \{2, 4, \ldots, p-1\}$ and all non-simple, non-projective, indecomposable Λ-modules $N_{j^\backprime}; j^\backprime = \{1, 2, \ldots, r\}$ are connected either ways by a sequence of Λ-module homomorphisms.

6. Let $N_1, N_2, N_3, N_4, N_5, N_6$ be any non-simple, non-projective, indecomposable Λ-modules. Hence,

$$N_6 \to P_j \to N_1,$$

$$N_1 \to V_j \to N_2,$$

$$N_3 \to P_e \to N_5,$$

and

$$N_4 \to N_3.$$

Then, all non-simple, non-projective, indecomposable Λ-modules are connected either ways by a sequence of Λ-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable Λ-modules in B_2 are connected either ways by a sequence of Λ-module homomorphisms as follows:

$$P_j \to N_{j^{\backprime}} \to V_j \to \ldots \leftarrow N^{\backprime}_{j^{\backprime}} \leftarrow V_e \leftarrow N^{\backprime\backprime}_{j^{\backprime}} \leftarrow P_e;$$

for all $j, e \in \{2, 4, \ldots, p-1\}$ and $j^{\backprime} = \{1, 2, \ldots, r\}$.
i.e. from definition (1.1.1), $\forall X^{\backprime}, Y^{\backprime} \in B_2$; there is a sequence of indecomposable modules $X^{\backprime} = X^{\backprime}_1, X^{\backprime}_2, \ldots, X^{\backprime}_t = Y^{\backprime}$ in B_2 such that for all $n \in \{1, 2, \ldots, t\}$ either

$$(X^{\backprime}_n, X^{\backprime}_{n+1})_\Lambda \neq 0 \quad \text{or} \quad (X^{\backprime}_{n+1}, X^{\backprime}_n)_\Lambda \neq 0.$$

Thus, The block B_2 does not split into union of pseudoblocks. So, B_2 is one pseudoblock.

Then, there are three blocks of Λ and three pseudoblocks of Λ in characteristic odd prime p.

Thus, For group algebra FG in characteristic odd prime number p the two notions blocks and pseudoblocks coincide.

□

EXAMPLE 5.5.2.
In group algebra *$FSL(2,p)$* ***at*** *$p = 2$; the representations of $SL(2,2) \cong S_3$ in characteristic 2; there exist two blocks and two pseudoblocks.*
Thus, for $FSL(2,2)$ in characteristic 2 the two notions blocks and pseudoblocks coincide; as stated in section (4.3).

In group algebra *$FSL(2,p)$* ***at*** *$p = 3$; let F be a field of characteristic 3, and let $SL(2,3)$ be a special linear group. Hence, the number of all indecomposable $FSL(2,3)$-modules are $p^2 - p + 1 = 7$.*

The simple $FSL(2,3)$-modules are V_1, V_2, and $V_3 \cong P_3$ (Steinberg module). The projective indecomposable $FSL(2,3)$-modules have the following structures:

$$\begin{array}{cc} P_1 & P_2 \\ & \\ V_1 & V_2 \\ | & | \\ V_1 & V_2 \\ | & | \\ V_1, & V_2, \end{array} \quad \textit{and } V_3 \cong P_3.$$

The non-simple, non-projective indecomposable $FSL(2,3)$-modules are

$$\begin{array}{c} V_1 \\ \Big| \\ V_1 \end{array} \quad \& \quad \begin{array}{c} V_2 \\ \Big| \\ V_2 \end{array}.$$

Hence, there are three blocks of $FSL(2,3)$ as follows:

$$B_1 = \left\{ V_1, P_1, \begin{array}{c} V_1 \\ \Big| \\ V_1 \end{array} \right\}, \quad B_2 = \left\{ V_2, P_2, \begin{array}{c} V_2 \\ \Big| \\ V_2 \end{array} \right\}, \text{ and } B_3 = \{V_3 \cong P_3\}.$$

Then, there are three pseudoblocks as follows:

The first pseudoblock is:

$$V_1 \to \begin{array}{c} V_1 \\ \Big| \\ V_1 \end{array} \to \begin{array}{c} V_1 \\ \Big| \\ V_1 \\ \Big| \\ V_1 \end{array}.$$

The second pseudoblock is:

$$V_2 \to \begin{array}{c} V_2 \\ \Big| \\ V_2 \end{array} \to \begin{array}{c} V_2 \\ \Big| \\ V_2 \\ \Big| \\ V_2 \end{array}.$$

And the third pseudoblock *contains the Steinberg module ($V_3 \cong P_3$). Hence, the blocks and pseudoblocks coincide for the group algebra $FSL(2,3)$ in characteristic 3.*

Now; we find the vertices of indecomposable $FSL(2,3)$-modules; the Sylow 3-subgroup is $U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \forall \lambda \in \mathbb{F}_3 \right\}$, where $|U| = 3$.
Then, the vertex of V_1, V_2 is Sylow 3-subgroup U, and the vertex of non-simple, non-projective indecomposable $FSL(2,3)$-modules is U from theorem (5.3.4).
Also, the vertex of each projective indecomposable $FSL(2,3)$-modules P_1, P_2, and P_3 is the trivial subgroup I from theorem (5.3.4).

In group algebra *$FSL(2,p)$* ***at*** *$p = 5$. From example (5.4.3), there are three pseudoblocks as follows:* ***The first pseudoblock is:***

$$
\begin{tikzcd}[column sep=small] & V_3 \arrow[dl, dash] \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\ V_1 & & V_3 & \end{tikzcd}
\to
\begin{tikzcd}[column sep=small] V_3 \arrow[dr, dash] & & V_1 \arrow[dl, dash] \\ & V_3 & \end{tikzcd}
\to
\begin{tikzcd}[column sep=small] & V_3 \arrow[dl, dash] \arrow[dr, dash] & \\ V_1 \arrow[dr, dash] & & V_3 \arrow[dl, dash] \\ & V_3 & \end{tikzcd}
$$

$$
\to
\begin{tikzcd}[column sep=small] & V_3 \arrow[dl, dash] \arrow[dr, dash] & \\ V_1 & & V_3 \end{tikzcd}
\to
\begin{array}{c} V_3 \\ | \\ V_3 \end{array}
\to V_3 \to
\begin{array}{c} V_1 \\ | \\ V_3 \end{array}
\to V_1 \to
\begin{array}{c} V_3 \\ | \\ V_1 \end{array}
\to
\begin{array}{c} V_1 \\ | \\ V_3 \\ | \\ V_1 \end{array}.
$$

The second pseudoblock is:

$$
\begin{tikzcd}[column sep=small] V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \arrow[dr, dash] & \\ & V_2 & & V_4 \end{tikzcd}
\to
\begin{tikzcd}[column sep=small] V_4 \arrow[dr, dash] & & V_2 \arrow[dl, dash] \\ & V_2 & \end{tikzcd}
\to
\begin{tikzcd}[column sep=small] & V_2 \arrow[dl, dash] \arrow[dr, dash] & \\ V_2 \arrow[dr, dash] & & V_4 \arrow[dl, dash] \\ & V_2 & \end{tikzcd}
$$

$$
\to
\begin{tikzcd}[column sep=small] & V_2 \arrow[dl, dash] \arrow[dr, dash] & \\ V_2 & & V_4 \end{tikzcd}
\to
\begin{array}{c} V_2 \\ | \\ V_2 \end{array}
\to V_2 \to
\begin{array}{c} V_4 \\ | \\ V_2 \end{array}
\to V_4 \to
\begin{array}{c} V_2 \\ | \\ V_4 \end{array}
\to
\begin{array}{c} V_4 \\ | \\ V_2 \\ | \\ V_4 \end{array}.
$$

The third pseudoblock *contains the Steinberg module ($V_5 \cong P_5$).*

Thus, there are three blocks of $FSL(2,5)$, and there are three pseudoblocks of $FSL(2,5)$. Hence, the blocks and pseudoblocks coincide for the group algebra $FSL(2,5)$ in characteristic 5.

PROPOSITION 5.5.3. *For the group algebra $\Lambda = FSL(2,p)$ in characteristic prime p, the block and pseudoblock notions coincide; i.e.*

$$Ind\Lambda / \underset{PS\Lambda}{\approx} \quad = \quad Ind\Lambda / \underset{\Lambda}{\approx}.$$

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